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The Open University of Sri Lanka
B.Sc. /B.Ed. Degree Programme
Final Examination - 2016/2017
Pure Mathematics - Level 05
PUU3143/PUE5143 – Riemann Integration

Duration: Two Hours

Date: 12.01.2018 **Time: 2.00 p.m. – 4.00 p.m.**

Answer Four Questions Only.

1. (a) Let f be a bounded function on $[a,b]$. If $m \le f(x) \le M$ for each $x \in [a,b]$, where $m, M \in \mathbb{R}$, prove that $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$ for each $P \in P[a,b]$.

   (b) Let $f(x) = \begin{cases} x, & x \in [0,2] \cap \mathbb{Q} \\ -x+2, & x \in [0,2] \cap \mathbb{Q}^c \end{cases}$. Show that $\overline{\int_0^2} f(x)dx = 3$.

   (c) Let $f(x) = k$ where k is a constant and $x \in [a,b]$.

   Show that f is Riemann integrable on $[a,b]$ and that $\int_a^b f(x)dx = k(b-a)$.

2. (a) Let f be a bounded function on $[a,b]$. For each $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$. Prove that f is Riemann integrable on $[a,b]$.

   (b) Let $f(x) = \cos x,\ x \in [0,\pi]$. Use Riemann's Criterion to show that f is Riemann integrable on $[0,\pi]$.

   (c) Let $f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q}, \\ 0, & x \in [0,1] \cap \mathbb{Q}^c. \end{cases}$

   Use Riemann's Criterion to show that f is not Riemann integrable on $[0,1]$.

3. (a) Let f be a bounded and monotonically increasing function on $[a,b]$.
   Prove that f is Riemann integrable on $[a,b]$.

   (b) Let $f(x) = \left(1 + \frac{1}{x}\right)^x, x \in [1,2]$. Show that f is Riemann integrable on $[1,2]$.

   (c) Let f be a bounded function on $[a,b]$. Prove that if f is continuous function on $[a,b]$, then f is Riemann integrable on $[a,b]$.

4. (a) Let $f(x)=\begin{cases} \frac{\sin x}{x}, & x\in(0,1], \\ 1, & x=0. \end{cases}$ Show that f is Riemann integrable on $[0,1]$.

(b) Let f be a bounded and Riemann integrable function on $[a,b]$ and k be a real number. Prove that kf is Riemann integrable on $[a,b]$ and

$\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

(c) Let f be a bounded function on $[a,b]$ and k be a non zero real number.

State whether the following statement is true or false:

If kf is Riemann integrable on $[a,b]$, then f is Riemann integrable on $[a,b]$.

Justify your answer.

5. (a) Let f and g be Riemann integrable on $[a,b]$. Prove that fg is Riemann integrable on $[a,b]$.

(b) Is the converse of Part (a) true? Justify your answer.

(c) Suppose that f is Riemann integrable on $[a,b]$ and for each $x\in[a,b]$, $f(x)\geq\delta$ for some $\delta>0$. Prove that $\frac{1}{f}$ is Riemann integrable on $[a,b]$.

6. (a) Let f be a continuous function on $[a,b]$ and g be a bounded and Riemann integrable function on $[a,b]$. Prove that $g(x)\geq 0$ on $[a,b]$, then there exists $c\in[a,b]$ such that

$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx$.

(b) Show that there exists $c\in[1,2]$ such that $\int_1^2 (x^7+x^3-2)\,(x^2+x+1)\,dx = \frac{29}{6}\left(c^7+c^3-2\right)$.

(c) Determine the convergence of each of the following improper integrals:

(i) $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$;

(ii) $\int_0^1 \frac{x}{1-x}\,dx$.