35 00059

The Open University of Sri Lanka
Department of Mathematics
B.Sc. / B.Ed. Degree Programme
Pure Mathematics – Level 05
Final Examination - 2016/2017

PUU3240 - Ring Theory & Field Theory

Duration: Three Hours

Date: 27.12.2017 **Time: 09.30a.m. – 12.30p.m.**

Answer Five Questions Only.

01. Consider the set of all integers $\mathbb{Z}$ together with the binary operations $\oplus$ and $\oplus$ defined by

$$a \oplus b = a + b - 1 \text{ for all } a, b \in \mathbb{Z}, \text{ and}$$
$$a \otimes b = ab - a - b + 2 \text{ for all } a, b \in \mathbb{Z}.$$

(i) Show that $\mathbb{Z}$ is commutative under the binary operations $\oplus$ and $\otimes$. [02Marks]

(ii) Show that distributive law hold under the binary operations $\oplus$ and $\otimes$. [04Marks]

(iii) Show that the additive identity $0_{\mathbb{Z}}$ is 1 and the multiplicative identity $1_{\mathbb{Z}}$ is 2 under the binary operations $\oplus$ and $\otimes$ respectively. [04 Marks]

(iv) Find the additive inverse for all $a \in \mathbb{Z}$. Justify your answer. [04Marks]

(v) Find the multiplicative identity for all $a \in \mathbb{Z} \setminus \{1\}$. Justify your answer. [04 Marks]

(vi) Solve the equation $x^2 \oplus 0 = 0_{\mathbb{Z}}$ in $(\mathbb{Z}, \oplus, \otimes)$. [02 Marks]

02. (a) Let $S = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{Z} \right\}$ and $R = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \middle| p, q, r, s \in \mathbb{Z} \right\}$.

(i) Show that S is a subring of R under the usual addition and multiplication. [04 Marks]

(ii) Is S an ideal of R? Justify your answer. [04 Marks]

(b) Define $f: S \longrightarrow \mathbb{Z}[\sqrt{2}]$ by $f\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = a + b\sqrt{2}$ for all $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \in S$, where

$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$

(i) Show that f is a homomorphism. [06 Marks]

(ii) Find the $ker f$. [03 Marks]

(iii) Show that $S \cong \mathbb{Z}[\sqrt{2}]$. [03 Marks]

03. (a) Let I and J be ideals of a ring R. Define a mapping $\phi: I \longrightarrow {}^{(I+J)}/_{J}$ satisfy the following conditions:

(i) ϕ is a onto ring homomorphism. [06 Marks]

(ii) $ker\phi = I \cap J$. [02 Marks]

(b) (i) **State** the Fundamental Theorem of Homomorphism for rings. [02 Marks]

(ii) Giving reasons, conclude that ${}^{I}/_{I \cap J} \cong {}^{(I+J)}/_{J}$, where I and J are ideals of a ring R. [05 Marks]

(iii) Verify that ${}^{(12)}/_{(84)} \cong {}^{(3)}/_{(21)}$. [05 Marks]

04. (a) Let $A = \{b + ra |\, b \in I \text{ and } r \in R\}$, where I is an ideal of a ring R and $a \in R$,

Show that:

(i) A is an ideal of R [04 Marks]

(ii) $I \subseteq A$ and $a \in A$. [02 Marks]

(iii) $A \subseteq \bigcap_{J \in B} J$, where $B = \{J \,|\, J \text{ is an ideal of } R \text{ containing } I \text{ and } a\}$

Deduce that $A = (I, a)$. [02 Marks]

(iv) If R is the set of integers $\mathbb{Z}$, then $(n_0) = \{n_0 r |\; r \in \mathbb{Z}\}$. [02 Marks]

(b) (i) Define a maximal ideal in a ring R. [02 Marks]

(ii) Let I be a proper ideal of a ring R. Prove that if $(I, a) = R$ for any $a \in R \setminus I$, then I is a maximal ideal of R. [04 Marks]

(iii) Let $I = (3)$. Show that $(I, 2) = \mathbb{Z}$. Deduce that (3) is a maximal ideal of $\mathbb{Z}$.

Is (12) a maximal ideal of $\mathbb{Z}$? Justify your answer. [04 Marks]

05\. (a) (i) Define a prime ideal in a ring R. [02 Marks]

(ii) Let I be a proper ideal of a ring R. Prove that I is a prime ideal if and only if the quotient ring $R/_I$ has no zero divisors. [06 Marks]

(b) Let $I = \{p(x) \in \mathbb{Z}[x] \mid p(0) = 0\}$, where $\mathbb{Z}[x]$ is the set of all polynomials over $\mathbb{Z}$.

(i) Show that I is an ideal in $\mathbb{Z}[x]$ [04 Marks]

(ii) Appling Fundamental Theorem of Homomorphism of rings for the mapping $\phi: \mathbb{Z}[x] \longrightarrow \mathbb{Z}$ given by $\phi(p(x)) = p(0)$ for all $p(x) \in \mathbb{Z}[x]$, show that $\mathbb{Z}[x]/_I \cong \mathbb{Z}$. Deduce that I is a prime ideal. [08 Marks]

06\. (a) Define each of the following:

(i) nil radical $\sqrt{I}$ of an ideal I of a ring R. [02 Marks]

(ii) semiprime ideal of a ring R. [02 Marks]

(b) Let I and J be two ideals of a ring R. Show that:

(i) $I \subseteq \sqrt{I}$ [04 Marks]

(ii) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. [08 Marks]

(iii) If $R/_I$ has no nilpotent element, then the ideal I is semiprime. [04 Marks]

07\. (a) (i) What is meant by a maximal principle ideal? [02 Marks]

(ii) Let R be a principle ideal domain. Show that every infinite chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \cdots$ of R has a maximal ideal. [04 Marks]

(b) (i) Define an irreducible element of a ring R. [02 Marks]

(ii) Show that, p is an irreducible element of an integral domain R if and only if (p) is a maximal principle ideal. [08 Marks]

(iii) Is $(2 + \sqrt{-5})$ a maximal principle ideal in $\mathbb{Z}(\sqrt{-5})$? Justify your answer. [04 Marks]

08. Prove or disprove each of the following:

(i) Boolean ring is commutative [04 Marks]

(ii) A finite ring with identity which has no zero divisors is a division ring. [04 Marks]

(iii) $\mathbb{Z}_p$ is a prime field for any prime $p \in \mathbb{Z}$. [04 Marks]

(iv) Every field contains only one prime field. [04 Marks]

(v) The quadratic domain $\mathbb{Z}(\sqrt{-5})$ is a unique factorization domain. [04 Marks]

****