00076

36

The Open University of Sri Lanka
B.Sc./ B.Ed. Degree Programme
Final Examination-2016/2017
Pure Mathematics- Level-05
PUU 3244 - Number Theory & Polynomials

Duration: Three Hours.

Date: 05-01-2018 **Time: 2.00p.m. To 5.00p.m.**

Answer Five questions only.
(State clearly any result that you used, without proof.)

(01). (a) Define each of the following:

(i) Inductive set.
(ii) Well ordered set.
(iii) Division Algorithm.

(b) Prove each of the following using Mathematical Induction:

(i) $\sum_{i=1}^{n} i!.i = (n-1)!-1$

(ii) Let $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, then $A^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$

(c) Show that the equation $n + 10 = 7$ has no solution in $\mathbb{N}$ where $\mathbb{N}$is the set of natural numbers.

(d) If $x, y \in \mathbb{R}$ and $m, n \in \mathbb{N}$ then prove each of the following

$(i)\ x^m.x^n = x^{m+n}$

$(ii)(x^m)^n = x^{mn}$

$(iii)(xy)^n = x^n y^n$

(02) (a) Prove that if $n \in \mathbb{N}$ there is no $m \in \mathbb{N}$ such that $n < m < n + 1$.

(You may use the following propositions without proof.
- If $m, n \in \mathbb{N}$ and $m > n$, then $m - n \in \mathbb{N}$.
- There is no natural number n such that $0 < n < 1$.)

(b) If $b \in \mathbb{Z}$ and $B = \{z \in \mathbb{Z}\,; z \geq b\}$ then prove that B is a well ordered set.

(You may use the following propositions without proof.
If T is a non-empty subset of $\mathbb{N}$, then T has a least element.)

(c) Prove each of the following:

(i) If $c|b$ and $b|a$ then $c|a$

(ii) If $b|a$ and $b|c$ then $b|(a \pm c)$

(iii)If $b|a$ and $c \in \mathbb{Z}$ then $b|ac$

(d) Show that $n^2 - n$ is an even number when n is an integer.

(03) (a) If S is a non-empty subset of $\mathbb{Z}$ such that,

$$s_1, s_2 \in S \Rightarrow s_1 + s_2 \in S \text{ and } s_1 - s_2 \in S.$$

Show that $S = \{0\}$ or S contains a least positive integer d such that $S = \{nd : n \in \mathbb{Z}\}$.

(b) Show that $(9^n - 1)$ is divisible by 8 for all integers.

(c) If $a, b, c \in \mathbb{Z} \setminus \{0\}$ then prove that $(a, b, c) = ((a, b), c)$; where (a, b) is the greatest common divisor of a, b and (a, b, c) indicates the same definition.

(d) Compute the greatest common divisor d, if $(7469, 2464, 132)$. Express d in the form $d = 7469a + 2464b + 132c$, where a, b and c are any integers.

(e) Find the least common multiple of 12012 and 1105.

(04) (a) If $a \equiv b \ (mod m)$ and $c \equiv d \ (mod m)$, then prove each of the following:

$$a + c \equiv b + d \ (mod m)$$

$$a - c \equiv b - d \ (mod m)$$

$$ac \equiv bd \ (mod m)$$

(b) Prove that for given $n \in \mathbb{Z}$ there exists a unique $r \in \mathbb{Z}_m$ such that $n \equiv r \ (mod m)$.

(c) If n is an odd integer then prove that $n^2 \equiv 1 \ (mod 4)$.

(d) Find the integer in $\mathbb{Z}_7$ to which $9 \times 13 \times 19 \times 400 \times 52$ is congruent modules 7.

(e) Prove or disprove that every linear congruence has a solution.

(5) (a) Let R be a commutative ring. If $f(x),\ g(x) \in R[x]$ and $g(x)$ is monic, then prove that there exists a unique $q(x),\ r(x) \in R[x]$ such that $f(x) = q(x)g(x) + r(x)$ with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

(b) If $f(x) = x^4 - x^3 - x^2 + 1$ and $g(x) = x^3 - 1$ are polynomials over $\mathbb{Q}[x]$ then find the greatest common divisor $d(x)$ of $f(x)$ and $g(x)$ and express it in the form $d(x) = f(x)u(x) + g(x)v(x)$ where $u(x),\ v(x) \in \mathbb{Q}[x]$.

(c) Let $\mathrm{R} = \{a + ib\sqrt{5} : a,\ b \in \mathbb{Z}\}$. Let $f(x) = 6x^2 + 2\sqrt{5}ix - 1$ and $g(x) = (1 + i\sqrt{5})x - 1$ be polynomials in $\mathrm{R}[x]$.

(i) Show that the units in R are ± 1.

(ii) Show that g is a divisor of f.

(ii) Show further that f and g do not have a greatest common divisor.

(6) (a) State and prove Eisentein's irreducibility criteria.

(b) Determine whether the polynomial $f(x) = 7x^4 + 12x^3 + 12x^2 + 6x + 1$ is irreducible in $\mathbb{Z}[x]$.

(c) Find all monic irreducible polynomials of degree 2 in $\mathbb{Z}_3[x]$.

(7) (a) (i) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and $n \geq 1$. If $\alpha \in \mathbb{Q}$ is a zero of $f(x)$ and $\alpha = \frac{r}{s}$ such that $(r, s) = 1$, then show that $r \mid a_0$ and $s \mid a_n$.

(ii) Find all rational roots of the polynomial $14x^4 - 51x^3 + 56x^2 - 21x + 2$ over $\mathbb{Q}$.

(b) (i) State Factor Theorem.

(ii) Find all factors of $f(x) = 3x^4 - 3x^3 - 6$ in $\mathbb{Q}[x]$, $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

(8) (a) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{C}[x]$, $a_n \neq 0$ and $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ are the zeros of $f(x)$ in $\mathbb{C}$.

Show that (i) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_0 S_{m-n} = 0$, if $m > n$,

(ii) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_{n-m+1} S_1 + m a_{n-m} = 0$, if $m \leq n$,

where $S_r = \sum_{i=0}^{n} \alpha_i^r$.

(b) If $a,\ b,\ c \in \mathbb{C}$ such that $a + b + c = 0$, then prove that,

$$6(a^5 + b^5 + c^5) = 5(a^2 + b^2 + c^2)(a^3 + b^3 + c^3).$$