The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2016/2017
Pure Mathematics - Level 03
PUU1140/PUE3140 –Logic & Mathematical Proofs

Duration: - Two hours

Date: - 04-08-2017 **Time: - 9:30am. – 11:30am.**

Answer 4 questions only

1. (a) Write down the contraposition of each of the following statements. Prove each of the following statements. Assume that $m, n \in \mathbb{N}$.

(i) If mn is odd then m is odd and n is odd.

(ii) If $m^2 + n^2 > 25$ then $m \geq 4$ or $n \geq 4$.

(iii) If the perimeter of a rectangle is 12 meters then the area of the rectangle is less than or equal to 9 square meters.

(b) Prove or disprove each of the following statements.

(i) There exists a non-zero real number r such that $\frac{r}{1+\sqrt{2}}$ is rational and $r(1+\sqrt{2})$ is rational.

(ii) For each $m.n \in \mathbb{N}$, $m^2 + n^2 > 50$ or $m^2 + n^2 < 50$.

2. (a) Write down the negation of each of the following statements. Do not place the phrase "It is not the case that" or the word "Not" in front of the statements. Prove each of the following statements.

(i) If $\pi + \sqrt{2}$ is a rational number then $\pi - \sqrt{2}$ is an irrational number.

(ii) $\pi 2^{\frac{1}{4}}$ is an irrational number or $\frac{2^{\frac{1}{4}}}{\pi}$ is an irrational number.

(iii) There exists a non-zero real number r such that $r\sqrt{2}$ is a rational number and $r + \sqrt{2}$ is a rational number.

(b) Let $p,\ q\ ,r$ be propositions. Prove each of the following statements.

(i) If $p \Rightarrow q$ is false then $(not(p \wedge q)) \ \Rightarrow q$ is false.

(ii) $(p \Rightarrow q) \wedge (p \Rightarrow r)$ is not a tautology.

3. (a) Prove that for each $m \in \mathbb{N}$, if $5+m$ is prime then $m = 2$ or m is not prime.

(b) Prove that for every $n \in \mathbb{N}$, $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$.

(c) Prove that for every $m,n \in \mathbb{Z}\setminus\{0\}$, $\frac{m}{n}+\frac{n}{m} \geq 2$ or $\frac{m}{n}+\frac{n}{m} \leq -2$.

(d) Prove that for each $m,n \in \mathbb{N}$, if $\sqrt{m}+\sqrt{n}$ is rational then $\sqrt{m}$ is rational and $\sqrt{n}$ is rational.

4. Prove each of the following statements.

(i) The sum of two prime numbers each larger than 2, is not a prime number.

(ii) There exists $n \in \mathbb{N}$ such that $\frac{n}{2015}+\frac{n^2}{2017}+\frac{n^3}{2019}$ is an integer.

(iii) For every $n \in \mathbb{N}$, $\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6} \in \mathbb{N}$.

(iv) For every $n \in \mathbb{N}$, $\frac{n^2+1}{3} \notin \mathbb{N}$.

5. For each of the following statements, state whether it is true or false and prove your answer.

(i) For each $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $m+n$ is not prime.

(ii) There exists $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $m+n$ is not prime.

(iii) There exist $m,n \in \mathbb{N}$ such that $\sqrt{m}+\sqrt{n}$ is irrational and $\sqrt{mn}$ is rational.

(iv) There exist $m,n \in \mathbb{N}$ such that $\sqrt{m}+\sqrt{n}$ is rational and $\sqrt{mn}$ is irrational.

6. (a) Prove that $\sqrt{2}$ is irrational.

(b) Let $n \in \mathbb{Z}$. Prove that $\sqrt{2^n}$ is irrational if and only if n is odd.

(c) Let $p, q \in \mathbb{Z}$. Prove that $\frac{p^2+q^2}{2}$ is odd if and only if p is odd and q is odd.

(d) Let $n \in \mathbb{N} \setminus \{1\}$. Prove that $\sqrt{2^n - 1}$ is irrational. Deduce that for each $n \in \mathbb{N}$ such that n is even, $\sqrt{1 - \frac{1}{2^n}}$ is irrational.