The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination - 2016/2017
Pure Mathematics- Level 04
PUU2143/PUE4143–Differentiable Functions

Duration: - Two hours

Date: - 20-01-2018 Time: - 1:30pm. – 3:30pm.

Answer **Four** questions only.

01. (i) (a) Let $f(x)=\begin{cases} x^3\sin(1/x), & x\neq 0 \\ 0 & ,x=0 \end{cases}$.

Use $\varepsilon-\delta$ definition to prove that f is differentiable at 0 and $f'(0)=0$.

(b) Let $g(x)=\begin{cases} x\sin(1/x), & x\neq 0 \\ 0 & ,x=0 \end{cases}$

Use $\varepsilon-\delta$ definition and the method of contradiction to prove that g is not differentiable at 0.

(ii) Construct a function h defined on $\mathbb{R}$ such that

(a) h is left differentiable at 0 and $h'_-(0)=2017$,

(b) h is right differentiable at 0 and $h'_+(0)=2018$.

Prove that any function h that satisfies (a) and (b) is continuous at 0.

02. (i) Let f be a function defined on $\mathbb{R}$ such that for each $x\in\mathbb{R}$,

$$x^2+3\leq f(x)\leq 2x^2-2x+4.$$

Compute $f'_+(1)$ and $f'_-(1)$ by showing all the work.

Is f differentiable at 1? Explain your answer.

(II) Let $f:\mathbb{R}\to\mathbb{R}$ be a function and c be a real number.

Prove that if f is differentiable at c, then $\lim_{n\to\infty}\{n[f(c+1/n)-f(c)]\}=f'(c)$.

Suppose f is a function and c is a real number such that $\lim_{n\to\infty}\{n[f(c+1/n)-f(c)]\}$ exists finitely. Does it follow that f is differentiable at c? Prove your answer.

03. (i) Let f be a function defined on an open interval (a,b) and $c\in(a,b)$. Assume that f is differentiable at c. Prove that if f has local maximum at c, then $f'(c)=0$. Is the converse true? Justify your answer.

(ii) Consider the function $g:\mathbb{R}\to\mathbb{R}$ given by

$$g(x)=\begin{cases} -(x-3)^2+3/2, & 5/2\le x \\ x-5/4, & 2\le x\le 5/2 \\ -5x/4+13/4, & 1\le x\le 2 \\ x+1, & x\le 1 \end{cases}.$$

Prove that (a) g is continuous on $\mathbb{R}$ by showing that g is continuous at $1, 2$ and $5/2$.

(b) g has a local maximum at 1 and g is not differentiable at 1.

(c) g has a local minimum at 2 and g is not differentiable at 2.

(d) g has a local maximum at 3, g is differentiable at 3 and $g'(3)=0$.

04. (i) Prove that there exists $c\in(0,1)$ such that $c^5=\dfrac{1+4c+9c^2+16c^3+25c^4}{90}$.

(ii) Let f be a function defined on $[0,1]$, continuous on $(0,1]$, differentiable on $(0,1)$ and $f(0)=f(1)$. Does it follow that there exists $c\in(0,1)$ such that $f'(c)=0$? Prove your answer.

(iii) Let g be a function defined on $[0,1]$, continuous on $[0,1]$, differentiable on $(0,1)$ and $g(0) \neq g(1)$. Does it follow that for each $c \in (0,1)$, $g'(c) \neq 0$? Prove your answer.

05. (i) Suppose f is defined on $[a,b]$, continuous on $[a,b]$ and differentiable on (a,b).

Prove that

(a) if $f'(x) > 0$ for each $x \in (a,b)$ then f is strictly increasing on $[a,b]$,

(b) if $f'(x) < 0$ for each $x \in (a,b)$ then f is strictly decreasing on $[a,b]$.

(ii) Let g be a function differentiable on (a,b) and $c \in (a,b)$. Let l be a real number such that $\lim_{x \to c} g'(x) = l$. Prove that $g'(c) = l$.

(iii) Let h be a function defined on $\mathbb{R}$ such that $\lim_{x \to 0} h'(x) = 2018$. Does it follow that h is differentiable at 0 and $h'(0) = 2018$? Prove your answer.

06. (i) Let f be a function continuous on $[a,b]$, differentiable on (a,b) and for each $x \in (a,b)$, $f'(x) \neq 0$. Prove that f is strictly increasing on $[a,b]$ or f is strictly decreasing on $[a,b]$.

(ii) Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \begin{cases} \dfrac{x+1}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$.

Does there exist a function $h : \mathbb{R} \to \mathbb{R}$ such that for each $x \in \mathbb{R}$, $h'(x) = g(x)$? Prove your answer.

(iii) Compute (a) $\lim_{x \to 0+} \left(\dfrac{1}{2x} - \dfrac{1}{\sin 3x} \right)$ (b) $\lim_{x \to 0-} \left(\dfrac{1}{2x} - \dfrac{1}{\sin 3x} \right)$. Show all the work.