The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2016/2017
Pure Mathematics – Level 04
PUU2144/PUE4144 – Group Theory I

Duration: - Two Hours

Date: -09. 01. 2018 **Time: - 9.30am. To 11.30am.**

Answer **FOUR** questions only.

1) a) Let $G = \{1,5,7,11\}$ and the operation $\otimes_{12}$ defined by $a \otimes_{12} b = r, 0 \le r < 12$ for all $a, b \in G$; where r is the remainder when ordinary multiplication ab is divided by 12.
   (i) By using composition table, show that $(G, \otimes_{12})$ is a group.
   (ii) Find the order of each element in G.
   (iii) Does G form a cyclic group under the operation $\otimes_{12}$? Justify your answer.

   b) Let H be a non-empty subset of a group G. Prove that $H \le G$, if and only if $ab^{-1} \in H, \forall\, a, b \in H$

2) a) Let $G = \mathbb{Z} \times \mathbb{Q}$, where $\mathbb{Z}$ is the set of integers and $\mathbb{Q}$ is the set of rational numbers. An operation o is defined on G by
   $(a,b)o(c,d) = (a + c, 2^c b + d)$ for $(a,b), (c,d) \in G$.
   Prove that (G, o) is a non-abelian group.

   b) Let G be the group of all non-zero complex numbers $a + bi$ under usual multiplication.
   (i) Find the inverse of $a + bi \in G$.
   (ii) Show that $H = \{a + bi \in G : a^2 + b^2 = 1\}$ is a sub group of G.

   c) Prove that every cyclic group is abelian.

3) Let G, G' be two groups and $f : G \to G'$ be a homomorphism.
   Define the Kernel of f (ker f).
   Prove that
   (i) Kernal of f is a normal subgroup of G.
   (ii) f is one to one if and only if $Ker\, f = \{e\}$, where e is the identity of G.
   (iii) $f(x^{-1}) = (f(x))^{-1}$ for all $x \in G$.
   (iv) $f(x) = f(y)$ if and only if $xy^{-1} \in ker\, f$ for $x, y \in G$.

4) a) Let G be a group and N be a normal subgroup of a group G.
Show that G/N is abelian if and only if, $xyx^{-1}y^{-1} \in N$ for all $x, y \in G$.

b) Let $G = \{(a,b): a,b \in \mathbb{R}, a \neq 0\}$ be the group under the operation $*$ defined by $(a,b) * (c,d) = (ac, bc + d)$ for all $(a,b),(c,d) \in G$.

(i) Show that $N = \{(1,b): b \in \mathbb{R}\}$ is a normal subgroup of G.
(ii) Use part (a) to show that G/N is abelian.

5) Let $(G,*)$ and (G', o) be two groups.
Define what is meant by an isomorphism $f : G \rightarrow G'$

a) Let G be the group of all non-zero complex numbers under multiplication. Let G' be the group of all real 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where both a and b are not zero under the matrix multiplication.
Show that $\phi : G \rightarrow G'$, defined by $\phi(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a homomorphism.
Is ϕ an isomorphism? Justify your answer.

b) Let G be a group and let $f : G \rightarrow G'$ such that $f(x) = x^{-1}$ be a homomorphism.
Show that G is abelian.

6) a) Let G be a group and H be a subgroup of G.
Show that if Ha and Hb are the two right cosets of G, then either $Ha \cap Hb = \emptyset$ or $Ha = Hb$.

b) If a and b are arbitrary distinct elements of a group G, and H is any subgroup of G, then show that

(i) $Ha = H = aH \Leftrightarrow a \in H$
(ii) $Ha = Hb \Leftrightarrow ab^{-1} \in H$
(iii) $aH = bH \Leftrightarrow b^{-1}a \in H$

c) Let H and K be two subgroups of a group G. Show that,

(iv) $H \cap K$ is a subgroup of G.
(v) If H and K are both normal in G, then $H \cap K$ is normal in K.