The Open University of Sri Lanka
B.Sc/B.Ed. DEGREE, CONTINUING EDUCATION PROGRAMME
Final Examination 2016/2017
Level 03 Pure Mathematics
PUU1142/PUE 3142– Vector Spaces

Duration: - Two hours

Date: - 29-12-2017 **Time: 2.00 p.m. to 4.00 p.m.**

Answer four questions only

1.

(a) Let V be a vector space over a field F. Using the axioms of a vector space, prove that

(i) $0 \cdot x = 0$ for all $x \in V$,

(ii) $\alpha \cdot 0 = 0$ for all $\alpha \in F$,

(iii) $(-\alpha) . x = -(\alpha . x)$, for all $\alpha \in F$ and $x \in V$.

(b) Let $V = \{ (a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$. For every $(a_1, a_2), (b_1, b_2) \in V$, define $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$ for $c \in \mathbb{C}$ where $\mathbb{C}$ is the complex number field. Is V a vector space over the field of complex numbers under these operations? Justify your answer.

(c) Let $V = \{ (a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$ For every $(a_1, a_2), (b_1, b_2) \in V$ define $(a_1, a_2) + (b_1, b_2) = (2a_1 + b_1, a_2 + 3b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$ for $c \in \mathbb{R}$ where $\mathbb{R}$ is the field of real numbers . Is V a vector space over the field of real numbers under these operations? Justify your answer.

2.

(a) Prove that if W_1 and W_2 are subspaces of a vector space V over a field F, then $W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1 \ and \ w_2 \in W_2 \}$ is a subspace of V over F.

(b) Let W_1 and W_2 are subspaces of the vector space V over the field F. Prove that if $W_1 \cup W_2$ is a subspace of vector space V over the field F then $W_1 \subset W_2$ or $W_2 \subset W_1$.

(c) Determine whether each of the following sets are subspace of the vector space $\mathbb{R}^2$ over the field $\mathbb{R}$ under usual addition and scalar multiplication :

(i) $A = \{(a + 2b, a + 1) \mid a, b \in \mathbb{R}\}$

(ii) $B = \{(a, a^2) \mid a \in \mathbb{R}\}$

3.

(a) Let U and V be vector spaces over a field F and $T: U \to V$ be a linear transformation. Prove that the kernel of T is a subspace of U.

(b) Let $M = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}\}$. Note that M is a vector space over the field $\mathbb{R}$ under the usual matrix addition and scalar multiplication.

Let the mapping $T : M \to M$ be defined by $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a + b & b \\ c & c + d \end{bmatrix}$.

Suppose $U = \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R}\}$.

(i) Show that T is a linear transformation,
(ii) Find the kernel of T.
(iii) Is U an invariant subspace of the vector space M over the field $\mathbb{R}$ under T? Prove your Answer.

4.

(a) Define an isomorphism between vector spaces U and V over a field F.

(b) Let U and V be vector spaces over a field F. Let $T: U \to V$ be a linear transformation onto V. Prove that T is an isomorphism if and only if

$ker\, T = \{0\}$, *where* 0 *is the additive identity of* U

(c) Define a basis of a vector space V over a field F.

(d) Let U and V be vector spaces over a field F, $T: U \to V$ be an isomorphism and $S = \{u_1, u_2, \ldots u_n\}$ be a basis of U. Prove that $T(S) = \{T(u_i) \mid u_i \in S\}$ is a basis of V.

5.

(a) Define linear independence and linear dependence of a non empty finite subset S of a vector space.

(b) Let $S = \{P_1 = 1 - x, P_2 = 5 + 3x - 2x^2, P_3 = 1 + 3x - x^2\}$ be a sub set of the vector space of all polynomials of degree at most 2 over $\mathbb{R}$. Is S linearly independent over the field $\mathbb{R}$? Justify your answer.

(c) Let U be a subspace of a vector space V over a field F, $T: U \to V$ be a linear transformation. Prove that $T(U) = \{T(u) | u \in U\}$ is a subspace of V.

(d) Let U be a subspace of a vector space V over a field F, $T: U \to V$ be a linear transformation and $S = \{u_1, u_2, \dots u_n\}$ be a linearly independent set of vectors in U. Is the set $T(S) = \{T(u_i) | u_i \in S\}$ always linearly independent? Justify your answer.

6.

(a) Let P_n be the vector space of polynomials of degree at most n over $\mathbb{R}$. In this space, with p and q arbitrary polynomials, we define

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx$$

(i) Show that P_n is a Euclidean space.

(ii) Find the lengths of the polynomials of $2x$ and $1-2x^2$ in P_2.

(iii) Find the distance between the polynomials $2x$ and $1-2x^2$ in P_2.

(b) Show that the three vectors $u_1 = (1, 2, 2)$, $u_2 = (1, -1, 2)$ and $u_3 = (1, 0, 1)$ form a basis for E^3, the usual Euclidean three space. Construct an orthonormal basis for E^3 out of $\{u_1, u_2, u_3\}$ using the Gram–Schmidt process.