The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination – 2019/2020
Pure Mathematics – Level 03
PEU3300/ PUU1140 – Mathematical Logic and Mathematical Proofs

Duration: Two hours.

Date: 09.01.2020	**Time: 09.30a.m. - 11.30a.m.**

Answer 4 questions only.

01. (a) Assumed $m,n \in \mathbb{N}$ and $x,y \in \mathbb{R}$. Write down the contraposition of each of the following statements. Prove each of the following statements.

(i) If mn is odd then m is odd and n is odd.

(ii) If $y^3 + yx^2 \leq x^3 + xy^2$ then $y \leq x$.

(b) Prove each of the following statements.

(i) If l,m,n are prime numbers greater than 2 then $m^3 + n^3 \neq l^3$.

(ii) If 5 is a factor of $4n+m$ then 5 is a factor of $4m+n$.

02. (a) Write down the negation of each of the following statements. Do not place the phrase "It is not the case that" or the word "Not" in front of the statements. Prove each of the following statements. Assume α and $\sqrt{3}$ are positive irrational numbers.

(i) $\alpha + \sqrt{3}$ is irrational or $\alpha - \sqrt{3}$ is irrational.

(ii) $\alpha\left(\sqrt{3}\right)^{1/2}$ is rational and $\dfrac{\left(\sqrt{3}\right)^{1/2}}{\alpha}$ is rational.

(iii) If $\alpha + \sqrt{3}$ is rational then $\alpha\sqrt{3}$ is irrational.

(b) Let p and q be statements. Find whether each of the following statements is true or false. Justify your answer without using truth tables or equivalent methods.

(i) $\left[\neg p \wedge (p \vee q)\right] \Rightarrow q$ is a tautology.

(ii) $(p \Rightarrow q) \wedge (\neg q \wedge p)$ is a tautology.

03. (a) Let $\langle x_n \rangle$ be a sequence of real numbers that satisfies $x_1 = 1$ and for each $n \in \mathbb{N}$, $x_{n+1} = \sqrt{1+3x_n}$. Prove that for each $n \in \mathbb{N}$, $x_n < 5$.

(b) Prove that for each $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that mn is not prime.

(c) Prove that for each $m, n \in \mathbb{N}$, if $\sqrt{\frac{m}{n}}$ is rational then $\sqrt{mn}$ is rational.

04. For each of the following statements, state whether it is true or false and prove your answer.

(a) There exists a real number x, there exists a real number y such that $x - y \in \mathbb{Q}$.

(b) For each real number x, for each real number y, $x - y \in \mathbb{Q}$.

(c) There exists a real number x such that for each real number y, $x - y \in \mathbb{Q}$.

(d) For each real number x, there exists a real number y such that $x \neq y$ and $x - y \in \mathbb{Q}$.

05. Prove or disprove each of the following statements.

(a) There exists a unique integer x such that $x^2 - 4x + 3 < 0$.

(b) There exists a unique real number x such that $x^2 - 4x + 3 < 0$.

(c) For any real number r, r^2 is irrational if and only if r is irrational.

(d) For any real numbers r, s, if r is non zero rational and s is irrational then rs is irrational.

06. (a) Prove that if p and q are positive integers then $\sqrt{pq} \leq \frac{p+q}{2}$.

(b) Let $p, q \in \mathbb{N}$. Prove that if p and q both odd or both even then $p^2 + q$ is divisible by 2.

(c) Let p and q be odd positive integers. Show that there exists $r \in \mathbb{Q}$ such that $rp \in \mathbb{N}$ and $r \geq \sqrt{q}$.