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The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme
Final Examination 2019/2020
Applied Mathematics – Level 04
ADU4302/APU2143 – Vector Calculus
Duration :- Two Hours.

Date : 31.12.2019 **Time:- 1.30 p.m. - 3.30 p.m.**

Answer Four Questions Only.

1. (a) State and sketch the domain of the function $f(x, y) = \ln(1 - x^2 - y^2)$.

   (b) Sketch the level curves of the function $f(x, y) = x^2 + y^2 - 4x - 6y + 13$.

   (c) Find the following limits, if they exist:

   (i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2 y}{x^2 + y^2}$, (ii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2 + y^2}$.

   (d) Discuss the continuity of the following function at the origin:

$$f(x, y) = \begin{cases} \dfrac{4x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

2. (a) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D. Explain briefly how you could determine its nature.

   (b) Find the maximum and minimum values of the function $f(x, y) = x^3 - 3xy + y^3$ and determine their nature.

   (c) Prove that the vector field $\underline{F} = (4xy - 3x^2 z^2)\underline{i} + 2x^2\,\underline{j} - 2x^3 z\underline{k}$ is conservative. Find the corresponding scalar potential function ϕ such that $\underline{F} = \underline{\nabla}\phi$.

3. (a) Prove that gradϕ is a vector normal to the contour surface $\phi(x, y, z) = c$, where c is a constant.

   (b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is given by $(x - x_0)\left(\dfrac{\partial F}{\partial x}\right)_P + (y - y_0)\left(\dfrac{\partial F}{\partial y}\right)_P + (z - z_0)\left(\dfrac{\partial F}{\partial z}\right)_P = 0$.

   (ii) Using the above result, find the equation of the tangent plane to the surface $xyz^2 = 1$ at the point $P(1, 1, 1)$.

(c) The function $T(x, y) = x^2 + y^2 - 3xy^3$ gives the temperature at each point (x, y) on the xy-plane. At the point $P\,(1, 2)$, in which direction should one go to get the most rapid increase in T?

4. (a) State Gauss' Divergence theorem in Vector Calculus.

(b) Verify the above theorem considering the vector field $\underline{F} = 2xy\underline{i} - yz\,\underline{j} + x^2\,\underline{k}$ and the surface S formed by the faces of the cube given by $0 \le x \le a$, $0 \le y \le a$ and $0 \le z \le a$.

5. (a) State Stokes' Theorem used in Vector Calculus.

(b) Verify Stokes' Theorem considering the vector field $\underline{F} = 2y\underline{i} - x\underline{j} + z\underline{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = 1;\ z \ge 0$.

(c) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ be the position vector of the point (x, y, z) and $r = |\underline{r}|$. Show that (for $r \ne 0$)

(i) $\underline{\nabla} r^2 = 2\underline{r}$, (ii) $\underline{\nabla}.\underline{r} = 3$, (iii) $\underline{\nabla} \times \underline{r} = \underline{0}$ (iv) $\nabla^2 r^2 = 6$.

where $\underline{\nabla}$ has a standard meaning.

6. (a) Suppose that S is a plane surface lying in the xy–plane and bounded by a closed curve C. If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$.

(b) Verify the above result for the integral $\oint_C \left(x^2 y dx + (y + xy^2)dy\right)$, where C is the closed curve of the region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$.