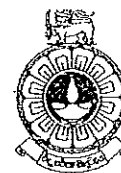


The Open University of Sri Lanka
 B.Sc./B.Ed Degree Programme
 Final Examination 2019/2020
 Applied Mathematics – Level 04
 ADU4302/APU2143 – Vector Calculus
 Duration :- Two Hours.



Date : 31.12.2019

Time:- 1.30 p.m. - 3.30 p.m.

Answer Four Questions Only.

1. (a) State and sketch the domain of the function $f(x, y) = \ln(1 - x^2 - y^2)$.

(b) Sketch the level curves of the function $f(x, y) = x^2 + y^2 - 4x - 6y + 13$.

(c) Find the following limits, if they exist:

$$(i) \lim_{(x,y) \rightarrow (0,0)} \frac{4x^2y}{x^2 + y^2}, \quad (ii) \lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2 + y^2}.$$

(d) Discuss the continuity of the following function at the origin:

$$f(x, y) = \begin{cases} \frac{4x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

2. (a) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D . Explain briefly how you could determine its nature.

(b) Find the maximum and minimum values of the function $f(x, y) = x^3 - 3xy + y^3$ and determine their nature.

(c) Prove that the vector field $\underline{F} = (4xy - 3x^2z^2)\underline{i} + 2x^2z\underline{j} - 2x^3z\underline{k}$ is conservative. Find the corresponding scalar potential function ϕ such that $\underline{F} = \nabla\phi$.

3. (a) Prove that $\text{grad } \phi$ is a vector normal to the contour surface $\phi(x, y, z) = c$, where c is a constant.

(b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point

$$P(x_0, y_0, z_0) \text{ is given by } (x - x_0) \left(\frac{\partial F}{\partial x} \right)_P + (y - y_0) \left(\frac{\partial F}{\partial y} \right)_P + (z - z_0) \left(\frac{\partial F}{\partial z} \right)_P = 0.$$

(ii) Using the above result, find the equation of the tangent plane to the surface $xyz^2 = 1$ at the point $P(1, 1, 1)$.

- (c) The function $T(x, y) = x^2 + y^2 - 3xy^3$ gives the temperature at each point (x, y) on the xy -plane. At the point $P(1, 2)$, in which direction should one go to get the most rapid increase in T ?

4. (a) State Gauss' Divergence theorem in Vector Calculus.

- (b) Verify the above theorem considering the vector field $\underline{F} = 2xy\underline{i} - yz\underline{j} + x^2\underline{k}$ and the surface S formed by the faces of the cube given by $0 \leq x \leq a$, $0 \leq y \leq a$ and $0 \leq z \leq a$.

5. (a) State Stokes' Theorem used in Vector Calculus.

- (b) Verify Stokes' Theorem considering the vector field $\underline{F} = 2y\underline{i} - x\underline{j} + z\underline{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = 1$; $z \geq 0$.

- (c) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ be the position vector of the point (x, y, z) and $r = |\underline{r}|$. Show that (for $r \neq 0$)

$$(i) \nabla r^2 = 2\underline{r}, \quad (ii) \nabla \cdot \underline{r} = 3, \quad (iii) \nabla \times \underline{r} = \underline{0} \quad (iv) \nabla^2 r^2 = 6.$$

where ∇ has a standard meaning.

6. (a) Suppose that S is a plane surface lying in the xy -plane and bounded by a closed curve C .

If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$.

- (b) Verify the above result for the integral $\oint_C (x^2 y dx + (y + xy^2) dy)$, where C is the closed curve of the region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$.