The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2019/2020
Pure Mathematics - Level 04
PEU4300/PUU2140– Real Analysis I

Duration: - Two Hours

Date: - 24-12-2019 **Time: - 1:30 p.m. – 3:30 p.m.**

Answer **Four** questions only.

1 (a) Prove that the sequence $\left\langle \frac{n-1}{\sqrt{10}-n} \right\rangle$ is bounded. Write down the greatest lower bound and the least upper bound. Guess the limit of this sequence and use the definition of a convergent sequence to prove that $\left\langle \frac{n-1}{\sqrt{10}-n} \right\rangle$ is convergent to your limit.

(b) Let $S_n = \sum_{k=3}^{n} \frac{1}{(k^2-4)}$ for $n \in \mathbb{N}$. Find S_n in terms of n and hence find $\lim_{n\to\infty} S_n$.

2. (a) Prove that if a sequence is monotonically increasing and bounded above, then it is convergent.

(b) The sequence $\langle u_n \rangle$ is defined recursively by $u_n = \sqrt{1+u_{n-1}}$ for $n \in \mathbb{N}$, with u_0 is a positive real number.

(i) Prove that if $u_0 = 1$, then $\langle u_n \rangle$ is monotonically increasing and bounded above. In this case find the limit of the sequence.

(ii) Prove that $\langle u_n \rangle$ is a constant sequence iff $u_0 = \frac{1+\sqrt{5}}{2}$.

3. (i) Let $u_n = \sum_{k=1}^{n} \frac{1}{(k-1)!}$. Prove that $u_n < 3$ for $n \in \mathbb{N}$.

(ii) Let $v_n = \frac{1}{\sqrt{2n^2+1}} + \frac{1}{\sqrt{2n^2+2}} + \frac{1}{\sqrt{2n^2+3}} + \ldots + \frac{1}{\sqrt{2n^2+n}}$ for $n \in \mathbb{N}$.

Show that $\frac{n}{\sqrt{2n^2+n}} \le v_n \le \frac{n}{\sqrt{2n^2+1}}$ for $n \in \mathbb{N}$. Deduce that $\lim_{n\to\infty} v_n = \frac{1}{\sqrt{2}}$.

(iii) Let $w_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{n+n-1} + \frac{1}{2n}$. Prove that $\langle w_n \rangle$ is convergent and $\frac{1}{2} \le \lim_{n\to\infty} w_n \le 1$.

4 (a) State the Cauchy criterion for a convergent series. Hence or otherwise prove that the Harmonic series $\sum \frac{1}{n}$ diverges.

(b) Find whether each of the following series is convergent. Justify your answer.

(i) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$,

(ii) $\sum_{n=1}^{\infty} p^n n^p$ (p is a real positive constant and $p > 1$),

(iii) $\sum_{n=1}^{\infty} \frac{2^n+(-2)^n}{(3^n+(-3)^n+1)}$,

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^n n 2^n}{3^n}$.

5 (a) Prove that the series $\sum_{n=1}^{\infty} \frac{1+(-1)^n n}{n^2}$ converges conditionally.

(b) Test each of the following series for absolute convergence:

(i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, (ii) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$,

(iii) $\sum_{n=1}^{\infty} \frac{-n+2}{n^3+1}$, (iv) $\sum_{n=1}^{\infty} x_n$, where $x_{2n} = -\frac{1}{n}$, $x_{2n-1} = \frac{1}{n}$.

6 (a) Find the lim sup and lim inf of each of the following sequences:

(i) $\left\langle 2\cos\left(\frac{n\pi}{3}\right)\right\rangle$, (ii) $\left\langle (-1)^n \left(1+\frac{1}{n}\right)\right\rangle$, (iii) $\left\langle \sum_{k=1}^{n} \frac{(-1)^k}{2^k}\right\rangle$.

(b) Find the radius of convergence of each of the following power series :

(i) $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$, (ii) $\sum_{n=1}^{\infty} a_n x^n$, where $a_n = \begin{cases} \frac{1}{2^n}, & n \text{ is even} \\ \frac{1}{n}, & n \text{ is odd} \end{cases}$.