The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination - 2019/2020
Pure Mathematics - Level 04
PEU4315/PUU2141/PUE4141 Continuous Functions

Duration: Two Hours

Date: 08.01.2020 **Time: 01.30pm - 03.30pm**

Answer <u>FOUR</u> Questions <u>ONLY.</u>

1. (a) Let $E = \left\{\frac{3n}{2n+1} : n \in \mathbb{N}\right\}$.Prove that $\frac{3}{2}$ is a limit point of E.

   (b) Let $f(x) = 2x+3, x \in [1,2)$. Prove that $\lim\limits_{x\to 2} f(x) = 7$ using the definition.

   (c) Let $g(x) = x^3, x \in \mathbb{R}$. Prove that $\lim\limits_{x\to 2} g(x) \neq 4$ using the definition.

2. (a) Let $f : (-\infty,3) \to \mathbb{R}$ be the function given by $f(x) = \sqrt{3-x}$, $x \in (-\infty,3)$.
   Prove that $\lim\limits_{x\to 3^-} f(x) = 0$.

   (b) Let $g: (-\infty, 3) \to \mathbb{R}$ be the function given by $g(x) = \sqrt{3-x} + 4,\ x \in (-\infty,3)$. Using above result in (a) deduce that $\lim\limits_{x\to 3^-} g(x) = 4$.

   (c) Find two functions f and g defined on $(-\infty, 3)$ such that $\lim\limits_{x\to 3^-} f(x)$ does not exist and $\lim\limits_{x\to 3^-} f(x)g(x) = 0$. Justify your answer.

3. (a) Use sandwich theorem to show that $\lim\limits_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$.

   (b) Prove that $\lim\limits_{x\to \infty} \frac{x^5+1}{x-2} = \infty$ using the definition.

   (c) Find two functions f and g such that $\lim\limits_{x\to -\infty} f(x) = \infty$, $\lim\limits_{x\to -\infty} g(x) = -\infty$ and $\lim\limits_{x\to -\infty} [f(x) + g(x)] = 2020$.

4. (a) Prove that the polynomial function $f(x) = x^3 + 5x - 7, x \in \mathbb{R}$ is continuous at 1.

   (b) Let $g(x) = \begin{cases} x^2 + 2, x \geq 1, \\ 2x + 1, x \leq 1. \end{cases}$ Prove that g is continuous at 1.

   (c) Find two functions f and g such that f, g are not continuous at 1 but $f + g$ is continuous at 1. Justify your answer.

5. (a) Let $f(x) = \begin{cases} x^2 + 1, x \leq 0, \\ 2x, \quad x > 0. \end{cases}$ Show that f is left-continuous at 0.

   (b) Let $g(x) = \begin{cases} x^3 + x + 2, & x \geq 3, \\ 15, & x < 3. \end{cases}$ Show that g is right-continuous at 3 and not continuous at 3.

   (c) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 30, & x > 1, \\ 14, & x \leq 1. \end{cases}$ and $g(x) = \begin{cases} 2020, & x = 30, \\ 11, & x > 30, \\ 6, & x < 30. \end{cases}$

   Find $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 30^+} g(x)$ and $\lim_{x \to 1^+} g(f(x))$ with justification.

6. (a) Let $f(x) = x^2, x \in (0,4)$. Prove that f is uniformly continuous on $(0,4)$.

   (b) Prove that the function $g(x) = x^2$ is continuous but not uniformly continuous on $(0, \infty)$.

   (c) Is the function $h(x) = \frac{x-1}{x+1}$ uniformly continuous on $[0,2]$? Justify your answer.