The Open University of Sri Lanka

B.Sc./B.Ed. Degree Programme

Final Examination-2019/2020

Pure Mathematics-Level 5

PUU3245 - Complex Analysis I

Duration: Three Hours

Date: 2020. 01.07 **Time: 1:30 p.m.- 4.30 p.m.**

Answer FIVE Questions Only.

1. (a) By considering eight roots of -1 , express z^8+1 as a product of four quadratic factors.

(b) Solve the equation $(1-z)^4 - z^4 = 0$.

(c) Let $0 \leq \theta < 2\pi$ and let $z_n = e^{in\theta}$, for $n \in \mathbb{N}$. Show that the sequence (z_n) is a Cauchy sequence only when $\theta = 0$.

{Hint: consider $\lim_{n\to\infty} |z_{n+1} - z_n|$}

2. (a) Let $f(z)$ be a function defined in $\mathbb{C}$. Let z_0 be a point in $\mathbb{C}$ .Write down the definitions of each of the followings:

(i) $f(z)$ is differentiable at $z_0 \in \mathbb{C}$.

(ii) $f(z)$ is analytic at $z_0 \in \mathbb{C}$.

(b) Let $f(z) = |z|^2$, for all $z \in \mathbb{C}$. Prove that $f(z)$ is differentiable only at 0. Is $f(z)$ analytic at 0 ? Justify your answer.

(c) Let $u(x,y) = 2x(1-y)$. Show that u is harmonic in $\mathbb{C}$ and find a harmonic conjugate v of u.

3. (a) Prove each of the following:

(i) $|\sin z|^2 = \sin^2 x + \sinh^2 y$, where $z = x + iy$ with $x, y \in \mathbb{R}$. Deduce that $\sin z$ is unbounded in $\mathbb{C}$.

(ii) $\cos z=0$ if and only if $z = (2n+1)\frac{\pi}{2}$, where n is an integer.

(b) Solve the equation $\sin z = 2$.

4. (a) State Cauchy's Integral Theorem and Cauchy's integral formula.

(b) Let C_1 and C_2 be two simple closed contours such that C_2 lying entirely inside C_1. If $f(z)$ is analytic in the set G consisting of points on and between C_1 and C_2, then by assuming Cauchy's Integral Theorem show that $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

(c) Evaluate $\int_C \frac{1}{z^2-1}dz$, where C is the circle $|z| = 25$ oriented counterclockwise.

(d) Evaluate $\int_C \frac{e^z}{z-3}dz$, where C is the circle $|z| = 5$ oriented counterclockwise.

5. (a) Determine the poles of each of the following functions and calculate the residue at each pole:

(i) $\frac{1}{z(1-z)^2}$, (ii) $\tan z$.

(b) Let $f(z) = \frac{1}{z(1-z^2)}$. Expand f in a Laurent series expansion of $f(z)$ in each of the following annuli:

(i) $0 < |z| < 1$,
(ii) $1 < |z-1| < 2$.

6. (a) Evaluate $\frac{2}{i}\int_C \frac{1}{5z^2+26z+5}dz$, where C is the circle with radius 1 centered at 0 oriented counterclockwise.

(b) Use the method of residues to show that $\int_0^{2\pi} \frac{1}{13+5\cos\theta}d\theta = \frac{\pi}{6}$.

7. (a) Find the Maclaurin Series (Taylor Series around 0) of $f(z) = \dfrac{z^2}{(1+z)^2}$.

(b) In each of the following, find and classify the singularities as poles (with order), removable singularities or essential singularities.
Justify your answers:

(i) $\dfrac{z+2}{z^3+z}$, (ii) $e^{\frac{1}{z}}$, (iii) $\dfrac{\sin z}{z}$, (iv) $\dfrac{1}{z}+\cos\dfrac{1}{z}$.

8. Use the method of residues to show that

(i) $\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2ab\,(a+b)}$, where a, $b > 0$.

(ii) $\int_{-\infty}^\infty \frac{1}{(x^2+1)^{n+1}}\,dx = \frac{(2n)!\pi}{2^{2n}\,(n!)^2}$, where n is a positive integer.