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The Open University of Sri Lanka
Department of Mathematics
B. Sc/ B. Ed Degree Programme
Final Examination - 2019/ 2020
Pure Mathematics– Level 05
PUU3242 – Combinatorics

DURATION: THREE HOURS

Date: 23 – 12 – 2019 **Time: 09.30 a.m. – 12.30 p.m.**

ANSWER FIVE QUESTIONS ONLY.

01. Using the method of *Mathematical Induction*, prove the *binomial theorem* for a positive integral index, by applying *Pascal's formula* at the induction step.

(a) Expand $(1+x)^5$ using the binomial expansion and give a *combinatorial reasoning* to obtain the corresponding coefficient by factorizing $(1+x)^5$ as $(1+x)(1+x)(1+x)(1+x)(1+x)$.

(b) Show that the sum of the coefficients of even powers of x in the expansion of $\left(1-3x+5x^2\right)^3$ is 378.

(c) Let $P(x)$ be a *multinomial expansion* of $\left(a+bx-cx^2\right)^5\left(a-bx+cx^2\right)^5$, where a, b and c are integers.

(i) Find the sum of the coefficients of $P(x)$ in terms of a, b and c.

(ii) Find the multinomial coefficient of x^7 in $P(x)$.

02. Find the total number of positive integers in each of the following cases:

(a) 2- digit numbers have their digits sum at most 6,
(b) 3-digit numbers with the middle digit 6 which are divisible by 3,
(c) Numbers which are divisors of 360,
(d) Numbers from 1 to 80 which are divisible by 2 or 5 but not by 4,
(e) Numbers can be formed from the digits 1, 2, 3, 4 and 5, if no digit is repeated.

03. 22 students follow advanced combinatorics course in their B. Sc degree programme at the Open University of Sri Lanka.

(a) Show that at least three students come from one of the nine provinces of Sri Lanka.

(b) The Lecturer of the course is preparing a multiple choice quiz (MCQ) paper and he wants to give each student the same set of questions but have the questions of each student appear in a different order. What is the least number of questions that the MCQ must contain?

(c) After the evaluation of the MCQ, the Lecturer found that the sum of their marks is 1325. Is it true that he can find 10 students in the class such that the sum of their marks is greater than 600?

(d) If the Lecturer assigns five grades based on their marks, then show that at least five students will receive the same grade.

(e) Show that there are at least 2 of the marks from all of the students whose sum is 160.

04. Let ${}^nC_r$ denote the number of r - object subsets of n objects and ${}^nP_r$ denote the number of r permutations of n objects. Prove that $r!\ {}^nC_r = {}^nP_r$.

(a) There are 2- *Grapes*, 1- *Orange*, 1- *Cherry*, 1- *Mango* and 1- *Apple* in a fruit plate.
In how many ways can a person pick two different fruits?

If 2- *Pears* are added to the above collection, in how many ways can the person pick two fruits?

(b) A password contains 6 to 8 characters long, where each character is a lowercase letter in the English alphabet or a digit. If each password must contain at least one digit, in how many ways can different passwords be created?

(c) How many different arrangements of the letters *A*, *B*, *C*, *D*, *E*, *M*, *N*, *R*, *S*, and *T* which contain either *CATS* or *MEN*.

(d) Ten contestants participate in a competition, the first place gets a house, the second place gets a motor car and the third place gets a motor bike, the next 2 places get equal amount of cash prizes. In how many ways can those prizes be awarded?

05. By defining the variables appropriately, model each of the following problems and **hence,** determine the solution.

(a) A fruit shop contains five types of fruits and fruits in each type are identical. In how many ways can seven fruits be chosen from the shop?

(b) From a collection of copies of a discrete mathematics book, copies of a modern physics book and copies of an organic chemistry book in a library, how many ways are there to select five books?

(c) Find the total number of positive integers less than 10,000, which have the sum of their digits equals 9.

(d) If each of three collectors get at least ten thousand rupees from 10 notes of five thousand rupees from a donator, find the number of ways of dividing those notes to the collectors.

(e) How many terms of the form $a^p b^q c^r d^s e^t$ from the multinomial expansion of $(a+b+c+d+e)^{15}$, where $p \geq 2$, $q \geq 3$, $r \geq 0$, $s \geq 3$ and $t \geq 2$ are positive integers.

06. The standard deck of 52 playing cards consists of four suits in two colors (red and black) and each suit contains different kinds which are in order: *A*, 2, 3, 4, 5, 6, 7, 8, 9, 10, *J*, *Q*, *K* (The last three are referred as face cards). A hand of a game is a selection of five cards from the deck. Find the probability of each of the following events that a hand contains:

(a) four cards of the same kind.
(b) two or more cards of the same kind.
(c) no *A*'s but have four cards of the same kind.
(d) only the face cards.
(e) in order, which are not the same suit.
(f) four *K*'s.
(g) at least two *K*'s.
(h) four red cards.

07. (a) Ten chocolates to be divided among four children, including twin brothers who are the youngest in a family. If each of the twins is to receive three chocolates and each of the others receives the same amount, in how many ways can those chocolates be divided?

(b) A boy wants to draw balls from a box which contains 12 identical balls. If he wants to select three balls first, then four balls and lastly the rest of the balls, find the number of different ways of drawing those balls.

(c) Find the number of permutations that can be formed from all the letters of the word "*MISSISSIPPI*". Out of these permutations:

(i) How many of those begin with an *S*?

(ii) How many of those begin and end with *I*?

(iii) How many of those have 2 *P*'s next to each other?

(d) Seven copies of a discrete mathematics book, five copies of a modern physics book, and eight copies of an organic chemistry book are in a book shop. In how many ways can five books be arranged on a shelf?

(e) Find the positive integers with the sum of digits of four digit numbers at most 34.

08. Let n, m and r be 3 positive integers such that $1 \le r \le n$ and $1 \le r \le m$. By using ONLY the *combinatorial arguments*, prove each of the following identities:

(a) $n \times {}^{n-1}C_{r-1} = r \times {}^{n}C_{r}$.

(b) ${}^{n}C_2 + {}^{m}C_2 + nm = {}^{n+m}C_2$.

(c) ${}^{n}C_0 + {}^{n}C_1 + \ldots + {}^{n}C_n = 2^n$.

(d) ${}^{n}C_r = {}^{n-2}C_r + 2\,{}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$.