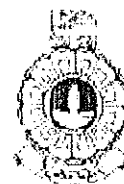


The Open University of Sri Lanka  
 B.Sc./B.Ed. Degree Programme  
 Final Examination – 2019/2020  
 Pure Mathematics- Level 05  
 PUU3141- Algebra of Complex Numbers



Duration: Two hours.

Date: 20.12.2019

Time: 09.30a.m- 11.30a.m

Answer 04 Questions only.

1. (a) Let  $z$  be a complex number. Prove each of the following:

(i)  $z\bar{z} = |z|^2$ .

(ii)  $z = -\bar{z}$  if and only if  $z = ri$  for some  $r \in \mathbb{R}$ .

(b) Let  $z_1$  and  $z_2$  be two complex numbers.

(i) Prove that  $|z_1 + z_2| \leq |z_1| + |z_2|$ .

Deduce that  $||z_1| - |z_2|| \leq |z_1 - z_2|$ .

(ii) Show that  $\left| \frac{z^4 + iz}{z^2 + z + 1} \right| \geq \frac{|z|^4 - |z|}{|z|^2 + |z| + 1}$  when  $|z| > 1$ .

2. (a) Let  $z_1$  and  $z_2$  be two non-zero complex numbers. Prove each of the following:

(i)  $\frac{z_1}{z_2}$  is a non-zero complex number.

(ii)  $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ .

(b) Express  $\frac{1+i}{\sqrt{3}-i}$  in polar form.

(c) (i) Let  $z_1 = z_2 = 2i$ . Verify that  $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$ .

(ii) Is it true that  $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$  for all  $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ ? Justify your answer.

3. (a) State De Moivre's Theorem for a positive integral index.

Let  $r > 0$  and  $\theta \in \mathbb{R}$ .

Given that  $z = r(\cos \theta + i \sin \theta)$  and  $w_k = \sqrt[4]{r} \left\{ \cos \left( \frac{\theta + 2k\pi}{4} \right) + i \sin \left( \frac{\theta + 2k\pi}{4} \right) \right\}$  for

$k = 0, 1, 2, 3$ , show that  $w_k^4 = z$  for  $k = 0, 1, 2, 3$ .

Hence, find the four solutions  $z_1, z_2, z_3, z_4$  of the equation  $z^4 = -8 + 8\sqrt{3}i$  in the Cartesian form.

(b) Let  $\omega = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$ .

Write down the value of  $\omega^5$  and prove that  $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ .

Simplify  $(\omega + \omega^4)(\omega^2 + \omega^3)$ .

Obtain a quadratic equation with integer coefficients having roots  $\omega + \omega^4$  and  $\omega^2 + \omega^3$ .

Hence, show that  $\cos \frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}$ .

4. (a) (i) Find all the complex roots of the equation  $\bar{z} = \frac{9}{z}$ .

(ii) Suppose that  $z_1$  and  $z_2$  are two complex numbers with  $z_1 z_2$  real and non-zero.

Show that there exists a real number  $r$  such that  $z_1 = \overline{r z_2}$ .

(b) (i) Let  $p(z) = 0$  be a non-constant complex polynomial with real coefficients. Prove that if  $\alpha$  is a root of  $p(z) = 0$ , then  $\bar{\alpha}$  is also a root of  $p(z) = 0$ .

(ii) Consider the complex quartic equation  $p(z) = z^4 + 3z^2 + 6z + 10 = 0$ .

Show that  $(1 - 2i)$  is a root of  $p(z) = 0$ .

Hence, find all the roots of  $p(z) = 0$ .

5. (a) Find all the complex numbers  $z$ , such that  $e^{iz} = 3$ .

(b) Prove that  $|e^z| = 1$  if and only if  $z$  is pure imaginary.

(c) (i) Solve the equation  $\cos z = 2$ .

(ii) Let  $z_1$  and  $z_2$  be two complex numbers such that  $\cos z_1 = \cos z_2$ .

Prove that there exists  $n \in \mathbb{Z}$  such that  $z_1 + z_2 = 2n\pi$  or  $z_1 - z_2 = 2n\pi$ .

6. (a) Let  $z_1$  and  $z_2$  be two non-zero complex numbers. Prove that

$$\log(z_1 z_2) = \log(z_1) + \log(z_2).$$

- (b) Show that  $\log\left[(1 + \sqrt{3}i)(1 - i)\right] = \log(1 + \sqrt{3}i) + \log(1 - i)$ .

- (c) Is it true  $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$  for all  $z_1, z_2 \in \mathbb{C}$ ? Justify your answer.

