The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination – 2019/2020
Pure Mathematics- Level 05
PUU3141- Algebra of Complex Numbers

Duration: Two hours.

Date: 20.12.2019	**Time: 09.30a.m- 11.30a.m**

Answer 04 Questions only.

1. (a) Let z be a complex number. Prove each of the following:

   (i) $z\bar{z}=|z|^2$.

   (ii) $z=-\bar{z}$ if and only if $z=ri$ for some $r\in\mathbb{R}$.

   (b) Let z_1 and z_2 be two complex numbers.

   (i) Prove that $|z_1+z_2|\le|z_1|+|z_2|$.

   Deduce that $\left||z_1|-|z_2|\right|\le|z_1-z_2|$.

   (ii) Show that $\left|\dfrac{z^4+iz}{z^2+z+1}\right|\ge\dfrac{|z|^4-|z|}{|z|^2+|z|+1}$ when $|z|>1$.

2. (a) Let z_1 and z_2 be two non-zero complex numbers. Prove each of the following:

   (i) $\dfrac{z_1}{z_2}$ is a non-zero complex number.

   (ii) $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$.

   (b) Express $\dfrac{1+i}{\sqrt{3}-i}$ in polar form.

   (c) (i) Let $z_1=z_2=2i$. Verify that $A\text{rg}\left(\dfrac{z_1}{z_2}\right)=A\text{rg}(z_1)-A\text{rg}(z_2)$.

   (ii) Is it true that $A\text{rg}\left(\dfrac{z_1}{z_2}\right)=A\text{rg}(z_1)-A\text{rg}(z_2)$ for all $z_1,z_2\in\mathbb{C}\setminus\{0\}$? Justify your answer.

3. (a) State De Moivre's Theorem for a positive integral index.
Let $r > 0$ and $\theta \in \mathbb{R}$.
Given that $z = r(\cos\theta + i\sin\theta)$ and $w_k = \sqrt[4]{r}\left\{\cos\left(\frac{\theta+2k\pi}{4}\right) + i\sin\left(\frac{\theta+2k\pi}{4}\right)\right\}$ for $k = 0,1,2,3$, show that ${w_k}^4 = z$ for $k = 0,1,2,3$.
Hence, find the four solutions z_1, z_2, z_3, z_4 of the equation $z^4 = -8 + 8\sqrt{3}i$ in the Cartesian form.

(b) Let $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.
Write down the value of ω^5 and prove that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.
Simplify $(\omega + \omega^4)(\omega^2 + \omega^3)$.
Obtain a quadratic equation with integer coefficients having roots $\omega + \omega^4$ and $\omega^2 + \omega^3$.
Hence, show that $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$.

4. (a) (i) Find all the complex roots of the equation $\bar{z} = \frac{9}{z}$.
(ii) Suppose that z_1 and z_2 are two complex numbers with $z_1 z_2$ real and non-zero. Show that there exists a real number r such that $z_1 = r\overline{z_2}$.

(b) (i) Let $p(z) = 0$ be a non-constant complex polynomial with real coefficients. Prove that if α is a root of $p(z) = 0$, then $\bar{\alpha}$ is also a root of $p(z) = 0$.
(ii) Consider the complex quartic equation $p(z) = z^4 + 3z^2 + 6z + 10 = 0$.
Show that $(1 - 2i)$ is a root of $p(z) = 0$.
Hence, find all the roots of $p(z) = 0$.

5. (a) Find all the complex numbers z, such that $e^{iz} = 3$.
(b) Prove that $|e^z| = 1$ if and only if z is pure imaginary.
(c) (i) Solve the equation $\cos z = 2$.
(ii) Let z_1 and z_2 be two complex numbers such that $\cos z_1 = \cos z_2$.

Prove that there exists $n \in \mathbb{Z}$ such that $z_1 + z_2 = 2n\pi$ or $z_1 - z_2 = 2n\pi$.

6. (a) Let z_1 and z_2 be two non-zero complex numbers. Prove that

   $\log(z_1 z_2) = \log(z_1) + \log(z_2)$.

   (b) Show that $\log\left[\left(1+\sqrt{3}i\right)(1-i)\right] = \log\left(1+\sqrt{3}i\right) + \log(1-i)$.

   (c) Is it true $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ for all $z_1, z_2 \in \mathbb{C}$? Justify your answer.