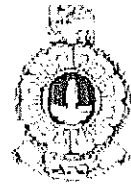


The Open University of Sri Lanka
 B.Sc./B.Ed. Degree Programme
 Final Examination – 2019/2020
 Pure Mathematics- Level 05
 PEU5304 – Introduction to Complex Analysis



Duration: Two hours.

Date: 20.12.2019

Time: 09.30a.m- 11.30a.m

Answer 04 Questions only.

1. (a) Let z be a complex number. Prove each of the following:

(i) $z\bar{z} = |z|^2$.

(ii) $z = -\bar{z}$ if and only if $z = ri$ for some $r \in \mathbb{R}$.

(b) Let z_1 and z_2 be two complex numbers.

(i) Prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.

Deduce that $||z_1| - |z_2|| \leq |z_1 - z_2|$.

(ii) Show that $\left| \frac{z^4 + iz}{z^2 + z + 1} \right| \geq \frac{|z|^4 - |z|}{|z|^2 + |z| + 1}$ when $|z| > 1$.

2. (a) Let z_1 and z_2 be two non-zero complex numbers. Prove each of the following:

(i) $\frac{z_1}{z_2}$ is a non-zero complex number.

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

(b) Express $\frac{1+i}{\sqrt{3}-i}$ in polar form.

(c) (i) Let $z_1 = z_2 = 2i$. Verify that $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$.

(ii) Is it true that $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$ for all $z_1, z_2 \in \mathbb{C} \setminus \{0\}$? Justify your answer.

3. (a) State De Moivre's Theorem for a positive integral index.

Let $r > 0$ and $\theta \in \mathbb{R}$.

Given that $z = r(\cos \theta + i \sin \theta)$ and $w_k = \sqrt[4]{r} \left\{ \cos \left(\frac{\theta + 2k\pi}{4} \right) + i \sin \left(\frac{\theta + 2k\pi}{4} \right) \right\}$ for

$k = 0, 1, 2, 3$, show that $w_k^4 = z$ for $k = 0, 1, 2, 3$.

Hence, find the four solutions z_1, z_2, z_3, z_4 of the equation $z^4 = -8 + 8\sqrt{3}i$ in the Cartesian form.

- (b) Let $\omega = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$.

Write down the value of ω^5 and prove that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

Simplify $(\omega + \omega^4)(\omega^2 + \omega^3)$.

Obtain a quadratic equation with integer coefficients having roots $\omega + \omega^4$ and $\omega^2 + \omega^3$.

Hence, show that $\cos \frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}$.

4. (a) (i) Find all the complex roots of the equation $\bar{z} = \frac{9}{z}$.

(ii) Suppose that z_1 and z_2 are two complex numbers with $z_1 z_2$ real and non-zero.

Show that there exists a real number r such that $z_1 = \overline{r z_2}$.

(b) (i) Let $p(z) = 0$ be a non-constant complex polynomial with real coefficients. Prove that if α is a root of $p(z) = 0$, then $\bar{\alpha}$ is also a root of $p(z) = 0$.

(ii) Consider the complex quartic equation $p(z) = z^4 + 3z^2 + 6z + 10 = 0$.

Show that $(1 - 2i)$ is a root of $p(z) = 0$.

Hence, find all the roots of $p(z) = 0$.

5. (a) Find all the complex numbers z , such that $e^{iz} = 3$.

(b) Prove that $|e^z| = 1$ if and only if z is pure imaginary.

(c) (i) Solve the equation $\cos z = 2$.

(ii) Let z_1 and z_2 be two complex numbers such that $\cos z_1 = \cos z_2$.

Prove that there exists $n \in \mathbb{Z}$ such that $z_1 + z_2 = 2n\pi$ or $z_1 - z_2 = 2n\pi$.

6. (a) Let z_1 and z_2 be two non-zero complex numbers. Prove that

$$\log(z_1 z_2) = \log(z_1) + \log(z_2).$$

(b) Show that $\log\left[(1 + \sqrt{3}i)(1 - i)\right] = \log(1 + \sqrt{3}i) + \log(1 - i)$.

(c) Is it true $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$ for all $z_1, z_2 \in \mathbb{C}$? Justify your answer.

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1
2

4
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1