00042

The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	: Mathematics
Level	: Five (05)
Name of the Examination	: Final Examination
Course Code and Title	**: PEU5300 – Riemann Integration**
Academic Year	: 2019/2020
Date	: 30.12.2019
Time	: 09.30 a.m. – 11.30 a.m.
Duration	: 2 hours
Index number	:

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **Six (06)** questions in **Two (02)** pages.
3. Answer any **Four (04)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary
5. Relevant log tables are provided where necessary.
6. Having any unauthorized documents/ mobile phones in your possession is a punishable offense
7. Use blue or black ink to answer the questions.
8. Circle the number of the questions you answered in the front cover of your answer script.
9. Clearly state your index number in your answer script

1) In this problem you may assume that for a bounded function f on $[a,b]$, and partitions P_1 and P_2 of $[a,b]$ such that P_2 is a refinement of P_1, the two results $L(P_1, f) \leq L(P_2, f)$ and $U(P_1, f) \geq U(P_2, f)$ hold.

(a) Let $f: [a, b] \to \mathbb{R}$ be a bounded function, and P, Q be arbitrary partitions of $[a, b]$. Prove that

(i) $L(P, f) \leq L(P * Q, f)$,

(ii) $U(P * Q, f) \leq U(Q, f)$, and

(iii) $L(P, f) \leq U(Q, f)$,

where $P * Q$ denotes the partition of $[a, b]$ given by $P_1 \cup P_2$.

(b) Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Suppose there exist sequences of partitions $(P_n)_{n=1}^{\infty}$ and $(Q_n)_{n=1}^{\infty}$ of $[a, b]$ such that the two sequences of numbers $\left(U(P_n, f)\right)_{n=1}^{\infty}$ and $\left(L(Q_n, f)\right)_{n=1}^{\infty}$ satisfy $\lim_{n\to\infty}\left(U(P_n, f) - L(Q_n, f)\right) = 0$. Assuming that $\lim_{n\to\infty} U(P_n, f)$ exists, show that f is Riemann integrable and $\int_a^b f = \lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(Q_n, f)$

2) Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Prove that if for each $\varepsilon > 0$, there exists a partition $P_\varepsilon \in P[a, b]$ such that $U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$ then f is Riemann integrable.

Using the above criterion, show that the function $f: [1, 3] \to \mathbb{R}$ defined by

$f(x) = \begin{cases} 1, & x \in [1, 2] \\ 3, & x \in (2, 3] \end{cases}$ is Riemann integrable on $[1, 3]$.

3)

(a) Let $f: [a, b] \to \mathbb{R}$ be a continuous function. Prove that f is Riemann integrable on $[a, b]$.

(b) Let $f: [1, 4] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3, & x \in [1,3] \\ 4, & x \in (3, 4] \end{cases}.$$

Show that the upper integral $\overline{\int_1^4} f(x)dx = 10$.

4) **(a)** Let $f: [a,b] \to \mathbb{R}$ be a Riemann integrable function and let $F_a: [a,b] \to \mathbb{R}$ be defined by $F_a(x) = \int_a^x f(t)dt$ for each $x \in [a,b]$. Prove that if f is continuous at $c \in (a,b)$, then f is differentiable at c, and $F_a{}'(c) = f(c)$.

(b) Let $f: [a,b] \to \mathbb{R}$ be a continuous function and let $F_b: [a,b] \to \mathbb{R}$ be defined by $F_b(x) = \int_x^b f(t)dt$ for each $x \in [a,b]$. Using the relation $\int_x^b f(t)dt = \int_a^b f(t)dt - \int_a^x f(t)dt$ show that $F_b(x)$ is differentiable at every point $c \in (a,b)$ and $F'_b(c) = -F'_a(c)$.

5) **(a)** Let $f: [a,b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a,b]$. Show that $L(P,f) \le S(P,f) \le U(P,f)$, where $S(P,f)$ is a Riemann sum for f corresponding to the partition P. Show that both these inequalities are strict for the function $f: [0,3] \to \mathbb{R}$ defined by $f(x) = \begin{cases} x^2 & x \in (0,3) \\ 1 & x = 0 \\ 3 & x = 3 \end{cases}$

and the partition $P = \{0,1,2,3\}$ of $[0,3]$ when $S(P,f)$ is calculated for points $\zeta_1 = \frac{1}{2}$, $\zeta_2 = \frac{3}{2}$ and $\zeta_3 = \frac{5}{2}$.

(b) Show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(3\left(\frac{2i}{n}\right) + 1\right)\left(\frac{2}{n}\right) = 8\,.$$

6) **(a)** Consider the following improper integrals:

$$\text{(i)} \int_0^2 \frac{1}{\sqrt{2-x}}\,dx, \qquad \text{(iii)} \int_{-2}^2 \frac{1}{x^{1/3}}\,dx$$

Explain why each of the above is an improper integral. Determine the convergence of each of the integrals and find the respective value if converges.

(b) Determine the convergence of $\int_1^\infty \frac{\cos^2 x}{\sqrt{x^5+1}}\,dx$ by using the Direct Comparison Test for improper integrals of bounded functions on unbounded intervals.

(c) Determine whether the integral $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ converges or diverges by using the Limit Comparison Test for improper integrals of unbounded functions on unbounded intervals.

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