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The Open University of Sri Lanka

B.Sc./B.Ed. Degree Programme

Final Examination-2019/2020

ADU5302/ADE5302/APU3143-Mathematical Methods

Applied Mathematics -Level 05

Duration: Two Hours.

Date: 19.12.2019	Time: 9.30 a.m.- 11.30a.m.

Answer FOUR questions only.

1. (a) Find the Laplace transform $L(t)$ of $te^{at}\sin at$.

   (b) Show that $L^{-1}\left\{\dfrac{1}{s^3(s+1)}\right\}=1-t+\dfrac{1}{2}t^2-e^{-t}$.

   (c) Using the convolution theorem, find the inverse Laplace transform of

   $$\left\{\frac{1}{(s^2+1)^3}\right\}.$$

   (d) Solve the following boundary value problem using the Laplace transform method:

   $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=e^{-x}\sin x \quad \text{where} \quad y(0)=0,\ y'(0)=1.$$

2.Obtain the formal expansion of the function f defined by $f(x)=1\ \left(1\le x\le e^{\pi}\right)$ as a series of orthonormal characteristic functions $\{\phi_n\}$ of the Sturm-Liouville problem

$$\frac{d}{dx}\left[x\frac{d}{dx}\right]+\frac{\lambda}{x}y=0$$
$$y'(1)=0$$
$$y'\left(e^{2\pi}\right)=0.$$

3.(a) Find the Fourier Series of $f(x)=\begin{cases}1, & -L\le x\le 0\\ L-x, & 0\le x\le L\end{cases}$ in the interval $-L\le x\le L$.

(b) Find the Fourier sine series and the Fourier cosine series of the following function f defined on $0\le x\le \pi$.

$$f(x)=\begin{cases}x & ;0\le x\le \pi/2\\ \pi-x & ;\pi/2\le x\le \pi.\end{cases}$$

4. (a) The Gamma function denoted by $\Gamma(p)$ corresponding to the parameter p is defined by the improper integral $\Gamma(p)=\int_0^\infty e^{-t}t^{p-1}dt, \quad (p>0)$.

Show that $\int_0^\infty \sqrt[4]{x}e^{-\sqrt{x}}\,dx=\frac{3}{2}\sqrt{\pi}$.

(You may assume that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.)

(b) The Beta function denoted by $\beta(p,q)$ is defined by

$\beta(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx,$ where p and q are positive parameters.

Using Gamma functions and Beta functions to evaluate each of the following integrals:

(i) $I=\int_0^1 x^{\frac{3}{2}}\left(1-x^2\right)^{\frac{5}{2}}dx.$

(ii) $\int_{-1}^{+1}(1+x)^{p-1}(1-x)^{q-1}dx.$

(iii) Showthat $I=\left[\int_0^\infty xe^{-x^8}dx\right]\times\left[\int_0^\infty x^2e^{-x^4}dx\right]=\frac{\pi}{16\sqrt{2}}.$

5. Let $J_p(x)$ be the Bessel function of order p given by the expansion

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p}.m!\Gamma(p+m+1)}.$$

Prove each of the following results:

(a) $\frac{d}{dx}\left\{x^p J_p(x)\right\} = x^p J_{p-1}(x).$

(b) $\frac{d}{dx}\left\{x^{-p} J_p(x)\right\} = -x^{-p} J_{p+1}(x).$

(c) $J_2'(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x).$

6. The Rodrigue's formula for the n^{th} degree Legendre polynomial denoted by $P_n(x)$ is given as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n.$$

$P_n(x)$ is also given by the sum

$$P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!} . x^{n-2m} \quad , \quad n = 0,\ 1,\ 2, \ldots ,$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$ whichever is an integer.

(a) Prove each of the following results:

(i) $xP_n'(x) = nP_n(x) + P_{n-1}'(x).$

(ii) $(2n+1)(1-x^2)P_n'(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)].$

(b) Express each of the following polynomials in terms of Legendre polynomials:

(i) $1 + x - x^2.$

(ii) $1 + 2x - 3x^2 + 4x^3.$