The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department : Mathematics
Level : 04
Name of the Examination : Final Examination
Course Title and - Code : Differentiable Functions – PEU4316 / PUU2143
Academic Year : 2019/20
Date : 20.02.2021
Time : 1.30 p.m. to 3.30 p.m.
Duration : Two Hours.

1. Read all instructions carefully before answering the questions.
2. This question paper consists of (6) questions in (3) pages.
3. Answer any (4) questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary.
6. Involvement in any activity that is considered as an exam offense will lead to punishment.
7. Use blue or black ink to answer the questions.
8. Clearly state your index number in your answer script.

1. (a). Let f be a real-valued function and c be a real number. Show that if f is differentiable at c, then f is continuous at c

State whether the converse of the above statement is true?

Justify your answer.

(b). Let g be a real-valued function defined by $g(x) = x^3, x \in \mathbb{R}$. Using $\varepsilon - \delta$ definition, prove that g is differentiable at $x = 1$ and $g'(1) = 3$.

(c). Let $h(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Show that h is differentiable only at $x \neq 0$.

2. (a). Let $f(x)$ and $g(x)$ be functions and c be a real number such that

$(c - \delta, c] \subseteq [Domn(f) \cap Domn(g)]$ for some $\delta_0 > 0$. Show that if $f(x)$ and $g(x)$ are left- differentiable at $x = c$, then $(f + g)(x)$ is left differentiable at $x = c$. Is the converse true? Justify your answer.

(b). Define $g: \mathbb{R} \longrightarrow \mathbb{R}$ by $g(x) = \begin{cases} x^3 - x + 1, & x \leq 1, \\ x^2, & x \geq 1. \end{cases}$

Show that g is left-differentiable at $x = 1$ and $g'_{-}(1) = 2$.

3. (a). State, without proof the Chain Rule

(b). Let $g(x) = \begin{cases} 2x^2, & x \in \mathbb{Q}, \\ 4x - 2, & x \in \mathbb{Q}^c, \end{cases}$ and $f(x) = \begin{cases} x^2 & ; x \in \mathbb{Q}, \\ 4x - 4 & ; x \in \mathbb{Q}^c. \end{cases}$

(i). Prove that g is differentiable at 1 and that $g'(1) = 4$.

(ii). Prove that f is differentiable at $g(1)$ and that $f'(g(1)) = 4$.

(iii). $h(x) = \begin{cases} 4x^4, & x \in \mathbb{Q}, \\ 16x - 12, & x \in \mathbb{Q}^c. \end{cases}$ Use the Chain Rule to show that h is differentiable at 1 and that $h'(1) = 16$.

4. (a) Let $f(x) = -x^2 + 6x - 5$. Show that $f(x)$ has a local maximum at $x = 3$.

(b) Suppose that g is a function defined on an open interval (a, b) and suppose that g has a local minimum at $c \in (a, b)$. Prove that if g is differentiable at c, then $g'(c) = 0$.

(c) Let $h(x) = x^2 + 4|x|$ for all $x \in \mathbb{R}$. Show that $h(0)$ is the minimum of h, but $h'(0)$ does not exist. Why does this not contradict the result in (b) above.

5. (a) State Rolle's theorem.
Let $f(x)$ be a differentiable on $[a, b]$ and twice differentiable on (a, b).
Suppose $f(a) = f'(a) = f(b) = 0$.
Show that there exist $x_0 \in (a, b)$ such that $f''(x_0) = 0$.

(b) State the Mean-Value theorem.
Let I be an open interval. Suppose that $f(x)$ and, $g(x)$ are functions such that $f'(x) = g'(x)$ for each $x \in I$.
Then prove that there exists a constant c such that $f(x) = g(x) + c$ for each $x \in I$.

(c) Using the Mean-Value theorem, prove that $|\sin x - \sin y| \leq |x - y|$ for each $x, y \in \mathbb{R}$.

6. (a) State the Intermediate – Value Property for Derivatives.

(b) Let $f(x) = \ln(1 + x)$.

(i) Calculate the fourth order Taylor polynomial $T_4(x)$ for $f(x)$ at $x = 0$.

(ii) Use Taylor's Theorem to write down a formula for the fourth remainder term $R_4(x)$, and deduce that

$$\frac{x^5}{5(1+x)^5} \leq f(x) - T_4(x) \leq \frac{x^5}{5} \text{ for all } x > 0.$$

(c) Show that the function g defined on $\mathbb{R}$ by
$g(x) = 5x^4 + 40x^3 + 90x^2 + 80x + 20$ for each $x \in \mathbb{R}$ has a relative minimum at $x = -1$.