The Open University of Sri Lanka
Department of Mathematics
B. Sc/ B. Ed Degree Programme
Final Examination - 2019/ 2020
Pure Mathematics– Level 04
PEU4301– Real Analysis II
Duration: Two Hours

Date: 20 -02- 2021 **Time: 01.30 p.m.-03.30 p.m.**

ANSWER FOUR (04) QUESTIONS ONLY

Q1)

(a) Let f be a function, $c \in \mathbb{R}$ such that $(c, \infty) \subseteq Domn(f)$, and $l \in \mathbb{R}$.

(i) State the definition of "$\lim_{x\to\infty} f(x) = l$".

(ii) Prove that if $\lim_{x\to\infty} f(x) = l$ exists for some $l \in \mathbb{R}$, then l is unique.

(b) Using the above defintion, prove that

$$\lim_{x\to\infty} \frac{5x^2+1}{8x^2+2x+1} = \frac{5}{8}.$$

(c) Let $f(x) = \begin{cases} 5x^2+1, & \text{if } x \geq 1 \\ 2x+4, & \text{if } x < 1 \end{cases}$. Show that f is continuous at $x = 1$.

Q2)

a) Let $f(x) = \frac{1}{x-1}$, $x \in \mathbb{R} \setminus \{1\}$. Find the composite function $f \circ f$ with the domain and determine the points of discontinuity, if any.

b) By clearly stating any results that you use, show that

$$\lim_{x\to 0^+} \sqrt{x}\, e^{\cos\left(\pi^2/x\right)} = 0.$$

c) Let $f: [2, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x+7}$. Show that f is uniformly continuous on $[2, \infty)$.

Q3)

a) State the Bolzano's Intermediate Value Theorem.

b) Use the above Theorem to show that there is a solution of the given equation between 1 and 2.

(i) $4x^4 - 5x^2 + 3x - 5 = 0$,

(ii) $\ln x = e^{-x}$ (you may assume that $e \approx 2.718$ and $\ln 2 \approx 0.693$).

c) Let $f(x) = \begin{cases} 3x+1, & \text{if } -1 \le x \le 1 \\ 4x^2, & \text{if } \ 1 < x \le 2 \end{cases}$.

(i) Prove that f is strictly increasing and continuous on $[-1,2]$.

(ii) Determine the image $f([-1,2])$ and find an explicit expression for $f^{-1}(x)$.

Q4)

a) Let f be a function and I be an interval. Define the meaning of expression that "f is differentiable on I" in terms of differentiability definition at a point.

Let $f\colon (-\infty, 1] \to \mathbb{R}$ be defined by $f(x) = x^3 - x + 1$. Show that f is differentiable on $(-\infty, 1]$.

b) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be functions and c be a real number such that both f and g are differentiable at point c. Prove that

(i) both $f+g$ and fg are differentiable at point c, and

(ii) $(f+g)'(c) = f'(c) + g'(c)$ and $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.

Q5)

a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function.

(i) If c is a real number such that f is differentiable at point c and $f'(c) > 0$ then prove that there exists $\delta > 0$ such that for each $x \in (c, c+\delta)$, $f(x) > f(c)$ and for each $x \in (c-\delta, \delta)$, $f(x) < f(c)$.

(ii) If f is differentiable and has a local maximum at $d \in \mathbb{R}$, then prove that $f'(d) = 0$. Clearly state any results that you use in this proof.

b) Let $f(x) = \begin{cases} 2x^2+1, & \text{if } x \ge 0 \\ 1, & \text{if } x < 0 \end{cases}$. Show that

(i) f is twice differentiable on $\mathbb{R} \setminus \{0\}$,

(ii) f is differentiable at point 0, and

(iii) f' is not differentiable at point 0, where, f' is the first derivative function of f.

Q6)

a) Let f be a function continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$. Prove that there exists $c \in (a,b)$ such that $f'(c) = 0$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 3x^4 + 5x^2 + 8x + 1$. Show that there exists, $c \in \mathbb{R}$ such that $f'(c) = 0$.

b) By applying L'Hospital's Rule, compute each of the following indterminate forms, if exists.

(i) $\lim\limits_{x \to 0^+} \frac{\ln(\sin x)}{\ln x}$ (ii) $\lim\limits_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$ (iii) $\lim\limits_{x \to \infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x$

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