The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination - 2020/2021
Pure Mathematics - Level 04
PEU4303/PUU2144 –Group Theory I

Date: 10-02-2021 **Time: 9.30 a.m. - 11.30 a.m.**

Answer <u>four</u> questions only.

Question (01)

Consider the following multiplication table (Cayley table) of group $G = \{I, R, R^2, F, FR, FR^2\}$ generated by two elements R and F.

$*$	I	R	R^2	F	FR	FR^2
I	I	R	R^2	F	FR	FR^2
R	R	R^2	I	FR^2	F	FR
R^2	R^2		R	FR	FR^2	F
F	F	FR	FR^2	I	R	
FR	FR	FR^2	F	R^2	I	
FR^2	FR^2		FR	R	R^2	I

Answer the following questions based on the above table.

(a) Find the following missing entries in the table. Justify your answer.

i) $R^2 * R$

ii) $FR^2 * R$

iii) $F * FR^2$

iv) $FR * FR^2$

(b) Find the order of each element of G.

(c) Simplify each of the following expressions and express it in one of the forms I, R, R^2, F, FR and FR^2. Show the relevant steps in your computations.

i) FR^3FR

ii) $FR^5F^{-7}R^{-1}F$

iii) $R^{2021}F^{-10}R^{-31}$

(d) Write down all the subgroups of G.

Question (02)

Consider the following groups

I. $(Z_6, +_6)$, where $Z_6 = \{0, 1, 2, 3, 4, 5, 6\}$ and $+_6$ is the addition modulo 6.

II. $(Z_9^{\times}, \times_9)$, where $Z_9^{\times} = \{1, 2, 4, 5, 7, 8\}$ and $\times_9$ is the multiplication modulo 9.

(a) Find the order of each element of the above groups.

(b) Find the inverse of each element of the above groups.

(c) Find all the subgroups of each of the above groups.

(d) Draw the lattice of subgrups for each of the above groups.

Question (03)

(a) Let $(G, *)$ be a group. Prove each of the following statements.

i) For every $a, b \in G$, $(a * b)^{-1} = b^{-1} * a^{-1}$.

ii) If G is abelian then for every $a, b \in G$, $(a * b)^4 = a^4 * b^4$.

iii) If for every $a, b \in G$, $(a * b)^2 = a^2 * b^2$, then G is abelian.

(b) Let $(G, *)$ be a group and $x \in G$. The **centralizer** of x is defined by

$$C(x) = \{z \in G : z * x = x * z\}.$$

Prove that $C(x)$ is a subgroup of G.

Question (04)

Let $G = \mathbb{Q} \setminus \{1\}$. Consider the binary operation $*: G \times G \to G$ given by $a * b = a + b - a \cdot b$, where "+", "−" and " · " are standard addition, subtraction & multiplication of rational numbers.

(a) Show that $*$ is an associative and commutative binary operation on G.

(b) Show that there is an identity for this binary operation.

(c) Show that every element of G has an inverse in G.

(d) Hence conclude that $(G, *)$ is an abelian group.

(e) Show that the subset $\{a \in \mathbb{Q}, a < 1\}$ is a subgroup of $(G, *)$.

(f) Is the subset $\{a \in \mathbb{Q}, a > 1\}$ a subgroup of $(G, *)$? Justify your answer.

Question (05)

Let $GL_2(\mathbb{R})$ be the set of all 2×2 real matrices with nonzero determinant. I.e,

$$GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} : p, q, r, s \in \mathbb{R} \text{ and } ps - qr \neq 0 \right\}$$

We know that $GL_2(\mathbb{R})$ is a group under the standard matrix multiplication. We denote this group by $(GL_2(\mathbb{R}), \times)$.

(a) Write down the identity element of $(GL_2(\mathbb{R}), \times)$.

(b) Write down the inverse of $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ in $(GL_2(\mathbb{R}), \times)$.

(c) Show that $(GL_2(\mathbb{R}), \times)$ is non-abelian.

Let $H = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x, y \in \mathbb{R} \text{ and } x \neq 0 \text{ or } y \neq 0 \right\}$.

(d) Prove that H is a sub group of $(GL_2(\mathbb{R}), \times)$.

(e) Determine the order of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

(f) Find an element of H which has order 8.

Question (06)

(a) State, without proof, the Lagrange's Theorem.

(b) Let G be a finite group and $g \in G$. Prove that the order of g divides $|G|$.

(c) Let G be a group and $|G| = p$, where p is a prime number.

i) Prove that G is a cyclic group.

ii) How many generators does G have? Justify your answer.

(d) Suppose G is a finite group with an element f of order 5 and an element g of order 8. Show that $|G| \geq 40$.