The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	: Mathematics
Level	: Five (05)
Name of the Examination	: Final Examination
Course Code and Title	**: APU3244 – Graph Theory**
Academic Year	: 2019/2020
Date	: 15.02.2021
Time	: 01.30 p.m. – 04.30 p.m.
Duration	: 3 hours
Index number	:

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **Eight (08)** questions in **Five (05)** pages.
3. Answer any **Five (05)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labeled diagrams where necessary.
5. Relevant log tables are provided where necessary.
6. Having any unauthorized documents/ mobile phones in your possession is a punishable offense.
7. Use blue or black ink to answer the questions.
8. Circle the number of the questions you answered in the front cover of your answer script.
9. Clearly state your index number in your answer script.

01. (a) Draw a simple graph to justify each of the following statements:

(i) A connected graph without having *Hamiltonian path.*

(ii) A *complete bipartite regular* graph of *degree* 2.

(iii) A *complete* graph that is a *wheel.*

(iv) A *regular* graph that is not *complete.*

(v) A *complete* graph that is *self-dual.*

(vi) A simple graph that is *self-complementary.*

(b) Let G be the simple graph with v *vertices* and e *edges*. Prove each of the following statements:

(i) If G is a *complete* graph, then it has $\frac{v(v-1)}{2}$ number of *edges*.

(ii) If G is a *regular* graph of degree r, then it has $\frac{vr}{2}$ number of *edges*.

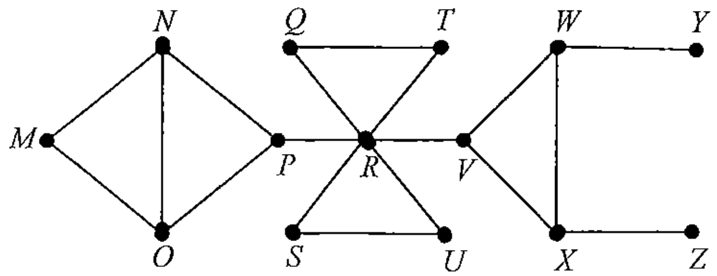
(iii) If M and m are the maximum and minimum *degrees* of the *vertices* of G respectively, then $m \leq \frac{2e}{v} \leq M$.

02. (a) (i) Draw the *line* graph, $L(K_4)$, of the *complete* graph K_4.

(ii) Draw the *total* graph, $T(K_3)$, of the *complete* graph K_3.

Are $L(K_4)$ and $T(K_3)$ *isomorphic*? Justify your answer.

(b) Let G be the following graph.

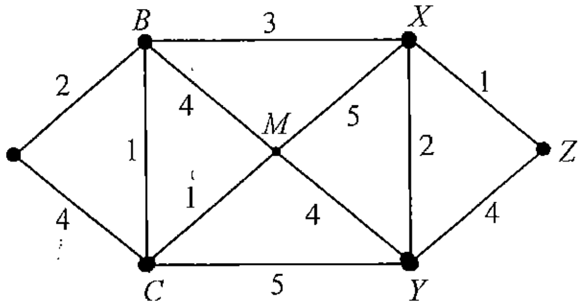


(i) Find all the *cut points* of G and draw the *cut point* graph.

(ii) Find all the *blocks* of G and draw the *block* graph.

(iii) Are there any *bridges* in G ? Justify your answer.

03. Let G be the following weighted graph.



(a) By choosing the *vertex* 'A' as the *root* of the tree and

(i) using the *depth first search algorithm*, produce a *spanning tree* of the maximum height.

(ii) using the *breadth first search algorithm*, produce a *spanning tree* of the minimum height.

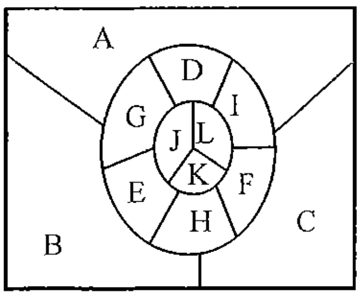
(b) By starting with the *vertex* 'A', apply the *Prim's algorithm* to find the minimum weighted *spanning tree* for G .

(c) Verify the result you obtained in part (b), by using *Kruskal's algorithm*.

(d) Apply an algorithm similar to *Kruskal's algorithm* to obtain a *spanning tree* of maximum weight for G .

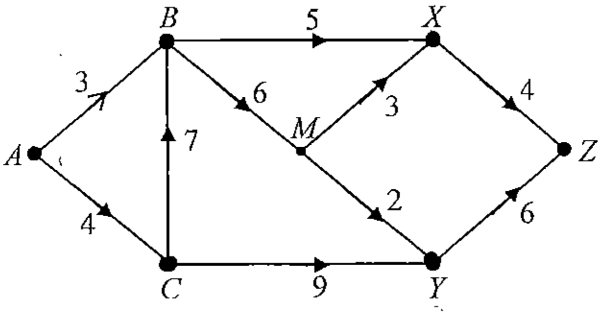
04. Define $k-colourable(f)$ and $k'-colourable(v)$ of a graph, where *f* represents any *face* in a map and *v* represents any *vertex* of a graph.

(a) (i) Find the value of k of the following map.



(ii) Find the value of k' of the *planar* graph corresponding to the above map.

(b) (i) Find the *critical path* from A to Z in the following *digraph D*.

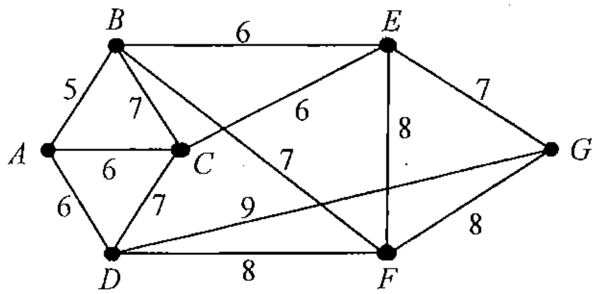


(ii) Verify the *Handshaking dilemma* for D.

(iii) Is D a *tournament*? Justify your answer.

05. Define a *semi-Eulerian path* and a *Hamiltonian path* of a graph.

The road development authority of a country located at A is paving the carpet on the roads of a district as given in the following map.



The number on each road in the map is the length of the road in kilometers, where A, B, C, D, E, F and G are the main cities in the district.

(a) A works engineer has started to inspect all the roads from A. Determine the minimum distance that the engineer has to travel to finish his inspection at G.

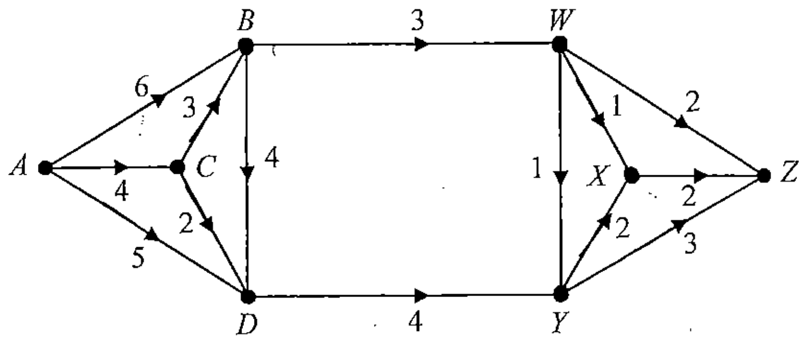
(b) A brick distributor from A has to unload the material in each of the main cities. Determine the minimum distance that the distributor has to travel to unload the material at G finally.

(c) Use the *Dijkstra's algorithm* to find the minimum distance from A to G.

Hence, find the total minimum distance of each of the works engineer and the distributor travel to return to A, after their duty, at the end of the day.

06. Let A and Z be any two distinct *vertices* of a *directed* graph N. State the *Menger's theorem* for N.

Let N be the following *network*. The number on each arc is the maximum units that can be sent from one destination to the other.



(a) (i) Write down all *arc-disjoint* paths and *vertex-disjoint paths* from A to Z in N.

(ii) Find a minimal *AZ-disconnecting* set and two *AZ-separating* sets in N.

Hence, verify the *Menger's theorem* for N.

(b) Draw a possible *flow* of N such that at least two units should be sent to each of the destinations B, C, and D from the *source* A.

Hence, verify the *maximum flow minimum cut theorem* for N.

07. (a) A group of nine girls participates in a 'drill display programme', which consists of four events, in a school sports-meet. Those nine girls stand each event in triples, three groups of three, so that in a particular time each pair of girls stands together in a group just once.

(i) Find the number of triples that can be formed.

(ii) Write down the different arrangements of girls in those four events.

(b) Let $E = \{1, 2, 3, 4, 5, 6, 7\}$ be a set of seven elements.

Let $B = \{234, 256, 271, 357, 361, 451, 467\}$ be a *family* of 3-element subsets of E.

Draw a *Fano matroid* (7 points plane) on the set E.

(c) In a group of singing contestants, three male singers M_1, M_2 and M_3 know four female singers F_1, F_2, F_3 and F_4 as follows. M_1 knows only F_1, F_3 and F_4; M_2 knows only F_2 and F_4; and M_3 knows only F_2 and F_3.

(i) Check the *marriage condition* for this problem

(ii) Find five different solutions to the above problem.

08. Let E be a non- empty finite set and let $S_1, S_2, ..., S_m$ be m non-empty subsets of E.

Define a *transversal* of a *family* $\mathfrak{I} = (S_1, \ldots, S_m)$.

(a) Let $E = \{1,2,3,4,5,6\}$ be a set of six elements. Let $S_1 = \{1,2,3\}, S_2 = \{1,2\}, S_3 = \{1,3\}$, $S_4 = \{2,3\}$ and $S_5 = \{2,4,6\}$ be five subsets of E.

(i) Show that the *family* $\mathfrak{I} = (S_1, S_2, S_3, S_4, S_5)$ has no *transversal.*

(ii) Determine whether subfamily $\mathfrak{I}' = (S_1, S_2, S_3, S_5)$ has *transversals* or not. Justify your answer.

(b) Write down the *incidence matrix* A of the *family* $\mathfrak{I}$. Hence,

(i) find the *term rank* of A.

(ii) verify the *Konig- Egervacy theorem* for A.

(iii) verify your result obtained in part (a)(i).