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The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	**: Mathematics**
Level	**: 05**
Name of the Examination	: Final Examination
Course Title and - Code	**: Number Theory & Polynomials - PUU 3244**
Academic Year	**: 2019/20**
Date	**: 26.10.2020**
Time	**: 1.30 p.m. To 4.30 p.m.**
Duration	**: Three Hours.**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of (8) questions in (4) pages.
3. Answer any ... (5) questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary
6. Involvement in any activity that is considered as an exam offense will lead to punishment
7. Use blue or black ink to answer the questions.
8. Clearly state your index number in your answer script

Note: For the proofs of Questions 1 to Question 4 you can use any proposition without proof.

1 (i) Prove or disprove each of the following:

(a) $(1,\infty)$ is an inductive set.

(b) $(-2,\infty)$ is an inductive set.

(ii) Show that the equation $n+12=10$ has no solution in $\mathbb{N}$.

(iii) If S is an inductive set and $S\subseteq\mathbb{N}$ then show that $S=\mathbb{N}$.

(iv) Prove each of the following by using Mathematical Induction.

(a) $\sum_{i=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

(b) $n! > 2^n$ for all $n \geq 4$ in $\mathbb{N}$.

2. (i) Prove that $\mathbb{Z}=\mathbb{N}\cup(-\mathbb{N})\cup\{0\}$.

(ii) Prove each of the following:

(a) 7^n+2 is divisible by 3 for all $n\in\mathbb{N}$.

(b) $3.5^{2n+1}+2^{3n+1}$ is divisible by 17 for all $n\in\mathbb{N}$.

(iii) Find the greatest common divisor g of 7500 and 723 and then find integers x and y to satisfy $7500x+723y=g$. Also find the least common multiple of 7500 and 325.

(iv) Compute $d=(1044, 1116, 1470)$ and express it in the form

$$d=1044a+1116b+1470c.$$

3. (i) Let $a,b,c,r\in\mathbb{Z}$ with $a\neq 0$ and $b\neq 0$.

Prove each of the following:

(a) $(ac,bc)=(a,b)c$ if $c>0$.

(b) $a\mid bc$ and $(a,b)=1 \Rightarrow a\mid c$.

(c) $a\mid c$, $b\mid c$ and $(a,b)=1 \Rightarrow ab\mid c$.

(d) $a=bq+r \Rightarrow (a,b)=(b,r)$

(ii) Show that there are infinitely many primes of the form $4k+3$.

4. (i) Let $a \equiv b \pmod m$ and $c \equiv d \pmod m$.

Prove each of the following:

(a) $a + c \equiv b + d \pmod m$

(b) $a - c \equiv b - d \pmod m$

(c) $ac \equiv bd \pmod m$

(ii) (a) Find the integer in $\mathbb{Z}_9$ to which $4 \times 8 \times 14 \times 403 \times 22$ is congruent modulo 9.

(b) Prove that $2^{12} \equiv 1 \pmod{63}$ and deduce that

$$2^{300} \equiv 1 \pmod{63} \text{ and}$$
$$2^{12} \equiv 1 \pmod{819}.$$

(iii) Prove that the linear congruence $35x \equiv 5 \pmod{14}$ has no solution in $\mathbb{Z}$.

(iv) If x_1 is a solution of $ax \equiv b \pmod m$ in $\mathbb{Z}$ and $x_2 \equiv x_1 \pmod m$ then prove that

x_2 is also a solution of $ax \equiv b \pmod m$ in $\mathbb{Z}$.

(5) (i) Let R be an integral domain. If $f(x),\ g(x) \in R[x]$ and $g(x)$ is monic, then prove that there exists a unique $q(x),\ r(x) \in R[x]$ such that $f(x) = q(x)g(x) + r(x)$ with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

(ii) If $f(x) = x^4 - x^3 - x^2 + 1$ and $g(x) = x^3 - 1$ are polynomials over $\mathbb{Q}[x]$ then find the greatest common divisor $d(x)$ of $f(x)$ and $g(x)$.

(6) (i) State and prove Eisentein's irreducibility criteria.

(ii) Determine whether the polynomial $f(x) = 8x^3 + 6x^2 - 9x + 24 \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$.

(iii) Discuss the irreducibility of $f(x) = 6x^2 + 3$ over $\mathbb{R}[x]$ and $\mathbb{Z}[x]$.

(7) (i) (a) Let $f(x)=\sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and $n \geq 1$. If $\alpha \in \mathbb{Q}$ is a zero of $f(x)$ and $\alpha = \frac{r}{s}$ such that

$(r, s) = 1$, then show that $r \mid a_0$ and $s \mid a_n$.

(b) Find all rational roots of the polynomial $f(x) = 36x^4 - 13x^2 + 1$ over $\mathbb{Q}$.

(ii) Find all factors of $f(x) = x^4 + 4$ in $\mathbb{Z}_5[x]$.

(8) (i) Let $f(x)=\sum_{i=0}^{n} a_i x^i \in \mathbb{C}[x]$, $a_n \neq 0$ and $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ are the zeros of $f(x)$ in $\mathbb{C}$.

Show that (a) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_0 S_{m-n} = 0$, if $m > n$,

(b) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_{n-m+1} S_1 + m a_{n-m} = 0$, if $m \leq n$,

where $S_r = \sum_{i=0}^{n} \alpha_i^r$.

(ii) If $a, b, c \in \mathbb{C}$ such that $a + b + c = 0$, then prove that,

$$6(a^5 + b^5 + c^5) = 5(a^2 + b^2 + c^2)(a^3 + b^3 + c^3).$$