The Open University of Sri Lanka

B.Sc/B.Ed. DEGREE, CONTINUING EDUCATION PROGRAMME

Final Examination 2020/2021

Level 03 Pure Mathematics

PEU3202 Vector Spaces

Duration: - Two Hours

Date: - 30-03-2022 **Time: 1.30 p.m. to 3.30 p.m.**

Answer four questions only

1.

(a) Suppose V is a vector space over a field F. Prove that for all $\alpha \in F$ and for all $x \in V-\{0\}$

(a) If $\alpha \cdot x = 0$ then $\alpha = 0$

(b) If $\alpha \cdot x = \beta \cdot x$ then $\alpha = \beta$

(b) Let $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a,b,c,d \in \mathbb{R} \right\}$. For every $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M$,

define $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$ and $\alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & 3\alpha d_1 \end{bmatrix}$

for $\alpha \in \mathbb{R}$, where $\mathbb{R}$ is the real number field. Is M a vector space over the field of real numbers under these operations? Justify your answer.

(c) Show that the three vectors $u_1 = (1, 2, 2)$, $u_2 = (1, -1, 2)$ and $u_3 = (1, 0, 1)$ form a basis of $\mathbb{R}^3$

2.

(a) Let V be a vector space over the field F and $W \subseteq V, W \neq \phi$. Show that W *is* a subspace of a vector space V over F if and only if for all $\alpha, \beta \in F$ and $x, y \in W$, $\alpha x + \beta y \in W$.

(b) Determine whether following sets are subspaces of the vector space $\mathbb{R}^3$ over the field $\mathbb{R}$ under usual addition and scalar multiplication in vector space $\mathbb{R}^3$. In each case justify your answer.

(i) $A = \{(a, b, c) \mid a, b, c \in \mathbb{R} \text{ and } b = 2a + a^2\}$

(ii) $B = \{(a, b, c) \mid a, b \in \mathbb{R} \text{ and } a + b = c\}$

(c) Suppose W_1 and W_2 are subspaces of a vector space V over a field F. Prove that $W_1 \cap W_2$ is a subspace of the vector space V over the field F.

3.

(a) Suppose V is a vector space over the field F. Show that if $\beta \in V$ is a linear combination of the set of vectors $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n \in V$, then the set $\{\beta, \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\}$ is linearly dependent.

(b) If α, β and γ are linearly independent vectors in V over a field F, prove that $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$ are also linearly independent.

(c) Suppose W is a subspace of a finite dimensional vector space V over the field F, then prove that $\dim W = \dim V$ if and only if $W = V$

4.

(a) Let $T : V \to W$ be a linear transformation. Show that

(i) $T(0) = 0$

(ii) $\ker T = \{0\}$ if and only if T is one to one.

(b) Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Note that V and W are vector spaces over the field $\mathbb{R}$ under the usual addition and scalar multiplication.

Consider the mapping $T : V \to W$ defined by $T(x, y) = (2x, x + y, x + 2y)$.

(i) Show that T is a linear transformation.

(ii) Find the kernel of T.

(iii) Is T an Isomorphism? Justify your answer.

5.

(a) Let $M = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in \mathbb{R} \}$. Note that M is a vector space over the field $\mathbb{R}$ under the usual matrix addition and scalar multiplication.

Let the mapping $T : M \to M$ be defined by $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a+b & b \\ 3c & d \end{bmatrix}$. Note that T is a linear Transformation

Determine whether the following sets are invariant subspaces of the vector space M over the field $\mathbb{R}$ under T

(i) $W = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a . b \in \mathbb{R} \right\}$

(ii) $W = \{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} | a, , c \in \mathbb{R} \}$

(b)

(i) Define an inner product space.

(ii) Let V be an inner product space over a field F. Prove that for $x_1, x_2, y_1, y_2 \in V$,

$$< x_1 + x_2, y_1 + y_2 > = < x_1, y_1 > + < x_1, y_2 > + < x_2, y_1 > + < x_2, y_2 >$$

(iii) Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$ where $u, v \in \mathbb{R}^3$.
Define $< u, v > = {x_1}^2 - {x_2}^2 - x_1 x_3$. Is $< u, v >$ an inner product on $\mathbb{R}^3$? Justify your answer.

6.

(a) Let u and v be any two vectors of a Euclidian Space.

(i) Prove that $\|u + v\| \leq \|u\| + \|v\|$

(ii) Define the angle between u and v

(iii) Suppose E^3 is the usual Euclidean three space and $u, v \in E^3$.
Let $u = (1, -1, 2)$ and $v = (2,1,0)$. Find the angle between u and v

(b) Show that the three vectors $u_1 = (1,1,1)$, $u_2 = (0,1,1)$ and $u_3 = (0,0,1)$ form a basis for E^3, the usual Euclidean three space. Construct an orthogonal basis for E^3 out of $\{u_1, u_2, u_3\}$ using the Gram–Schmidt process.