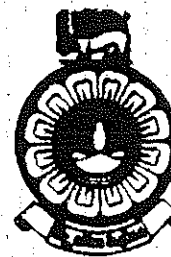


The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme



Department	: Mathematics
Level	: Level 04
Name of the Examination	: Final Examination
Course Code and Title	: PEU4301 - Real Analysis II
Academic Year	: 2020/ 2021
Date	: 01.04.2022
Time	: 02.00 p.m.-04.00 p.m.

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **six (06)** questions in **two (02)** pages.
3. Answer any **four (04)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Having any unauthorized documents/ mobile phones in your possession is a punishable offense.
6. Use blue or black ink to answer the questions.
7. Circle the number of the questions you answered in the front cover of your answer script.
8. Clearly state your index number in your answer script.

Q1)

(a) Let $f(x) = \frac{2x+1}{3x^2+2}$, $x \in \mathbb{R}$.

By using the definition of the limit of a function at a point, show that

$$\lim_{x \rightarrow 1} f(x) = \frac{3}{5}.$$

(b) Let $g(x) = \frac{1}{(x-10)^2}$, $x \in \mathbb{R} \setminus \{10\}$.

By using the definition of the infinite limit $+\infty$ of a function at a point, show that $\lim_{x \rightarrow 10} g(x) = +\infty$.

(c) Let $h(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ \sqrt{x}, & \text{if } x < 1 \end{cases}$.

By using $\varepsilon - \delta$ method, show that h is both left-continuous and right-continuous at $x = 1$.

Q2)

(a) Let $f(x) = 2x^2 + 1$, $x \in \mathbb{R}$.

By using $\varepsilon - \delta$ definition of continuity, prove that f is continuous at any point $c \in \mathbb{R}$.(b) Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} \sqrt{-x}, & x \in (-\infty, 0) \\ 1, & x = 0 \\ \sqrt{x} \sin\left(\frac{1}{x}\right), & x \in (0, +\infty) \end{cases}$$

Show that $\lim_{x \rightarrow 0^-} f(x) = 0 = \lim_{x \rightarrow 0^+} f(x)$.Is f continuous at $x = 0$? Justify your answer.

Q3)

(a) Let f and g be functions defined on \mathbb{R} . If f and g are continuous at $c \in \mathbb{R}$, prove that $f + g$ is continuous at c by using the $\varepsilon - \delta$ definition of continuity.(b) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \frac{x^2}{x^2+1}$. Prove that f is uniformly continuous on \mathbb{R} .(c) By using the Sandwich Theorem, find $\lim_{x \rightarrow 0} x^2 \left(1 + \sin\left(\frac{1}{x}\right)\right)$.

Q4)

(a) Let $f(x) = \frac{2x+3}{x+1}$, $x \in \mathbb{R}$.

By using the definition of the derivative of a function at a point, find $f'(0)$.

(b) Let $f(x) = \begin{cases} \frac{x}{1+x} & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$

Prove that the derivative of f does not exist at $x = 0$.

Q5)

(a) State the Mean-Value Theorem for Derivatives.

Using the above Theorem or otherwise show that for $a, b \in \mathbb{R}$ such that $0 < a < b$,

$$\sqrt{b} - \sqrt{a} < \frac{b-a}{2\sqrt{a}}.$$

(b) By using Bolzano's Intermediate Value Theorem show that the equation $\sin(10x) = x^2 - 1$ has a solution in the interval $(0, 3)$.

Q6)

(a) Let g be a three times differentiable function on $[a, b]$ and $g(a) = g'(a) = g(b) = g'(b) = 0$. Show that there exists $c \in (a, b)$ such that $g'''(c) = 0$.

(b) By applying L'Hospital's Rule, compute the limit (if exists) of each of the following indeterminate forms.

(i) $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^2}$ (ii) $\lim_{x \rightarrow 1} \left(\frac{1}{x-1} - \frac{\ln x}{(x-1)^2} \right)$ (iii) $\lim_{x \rightarrow \infty} \left(1 - \frac{5}{7x} \right)^{2x}$

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Handwritten text at the top of the page, possibly a title or introductory sentence.

$$f(x) = \frac{1}{x^2} \quad (x > 0)$$

Handwritten text below the first equation, possibly a definition or property.

Handwritten text in the middle section, possibly a derivation or explanation.

$$\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$$

Handwritten text below the second equation, possibly a conclusion or further notes.

Handwritten text in the lower middle section, possibly a final statement or example.

Handwritten text in the lower section, possibly a summary or additional remarks.

Handwritten text in the bottom section, possibly a signature or date.