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The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	: **Mathematics**
Level	: **Level 04**
Name of the Examination	: **Final Examination**
Course Code and Title	: **PEU4301 - Real Analysis II**
Academic Year	: **2020/ 2021**
Date	: **01.04.2022**
Time	: **02.00 p.m.-04.00 p.m.**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **six (06)** questions in **two (02)** pages.
3. Answer any **four (04)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Having any unauthorized documents/ mobile phones in your possession is a punishable offense.
6. Use blue or black ink to answer the questions.
7. Circle the number of the questions you answered in the front cover of youranswer script.
8. Clearly state your index number in your answer script.

Q1)

(a) Let $f(x) = \frac{2x+1}{3x^2+2}, x \in \mathbb{R}$.
By using the definition of the limit of a function at a point, show that $\lim_{x\to 1} f(x) = \frac{3}{5}$.

(b) Let $g(x) = \frac{1}{(x-10)^2}, x \in \mathbb{R} \setminus \{10\}$.
By using the definition of the infinite limit $+\infty$ of a function at a point, show that $\lim_{x\to 10} g(x) = +\infty$.

(c) Let $h(x) = \begin{cases} x^2, & \text{if } x \le 1 \\ \sqrt{x}, & \text{if } x < 1 \end{cases}$.
By using $\varepsilon - \delta$ method, show that h is both left-continuous and right-continuous at $x = 1$.

Q2)

(a) Let $f(x) = 2x^2 + 1, x \in \mathbb{R}$.
By using $\varepsilon - \delta$ definition of continuity, prove that f is continuous at any point $c \in \mathbb{R}$.

(b) Consider the function $f: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sqrt{-x}, & x \in (-\infty, 0) \\ 1, & x = 0 \\ \sqrt{x}\sin\left(\frac{1}{x}\right), & x \in (0, +\infty) \end{cases}$$

Show that $\lim_{x\to 0^-} f(x) = 0 = \lim_{x\to 0^+} f(x)$.

Is f continuous at $x = 0$? Justify your answer.

Q3)

(a) Let f and g be functions defined on $\mathbb{R}$. If f and g are continuous at $c \in \mathbb{R}$, prove that $f + g$ is continuous at c by using the $\varepsilon - \delta$ definition of continuity.

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x^2}{x^2+1}$. Prove that f is uniformly continuous on $\mathbb{R}$.

(c) By using the Sandwich Theorem, find $\lim_{x\to 0} x^2 \left(1 + \sin\left(\frac{1}{x}\right)\right)$.

Q4)

(a) Let $f(x) = \frac{2x+3}{x+1}, x \in \mathbb{R}$.
By using the definition of the derivative of a function at a point, find $f'(0)$.

(b) Let $f(x) = \begin{cases} \frac{x}{1+x}, & \text{if } x \geq 0 \\ x^2, & \text{if } x < 0 \end{cases}$

Prove that the derivative of f does not exist at $x = 0$.

Q5)

(a) State the Mean-Value Theorem for Derivatives.
Using the above Theorem or otherwise show that for $a, b \in \mathbb{R}$ such that$0 < a < b$,

$$\sqrt{b} - \sqrt{a} < \frac{b-a}{2\sqrt{a}}.$$

(b) By using Bolzano's Intermediate Value Theorem show that the equation $\sin(10x) = x^2 - 1$ has a solution in the interval $(0, 3)$.

Q6)

(a) Let g be a three times differentiable function on $[a, b]$and $g(a) = g'(a) = g(b) = g'(b) = 0$. Show that there exists $c \in (a, b)$ such that $g'''(c) = 0$.
(b) By applying L'Hospital's Rule, compute the limit (if exists) of each of the following indterminate forms.

(i) $\lim_{x \to 0} \frac{\sin x - x}{x^2}$ (ii) $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{\ln x}{(x-1)^2}\right)$ (iii) $\lim_{x \to \infty} \left(1 - \frac{5}{7x}\right)^{2x}$

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