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00290

The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc. / B. Ed Degree Programme

Department	**: Mathematics**
Level	**: 04**
Name of the Examination	**: Final Examination**
Course Title and - Code	**: Vector Calculus – ADU4302**
Academic Year	**: 2020/21**
Date	**: 13.12.2021**
Time	**: 1.30 p.m. To 3.30 p.m.**
Duration	**: Two Hours.**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **(6)** questions in **(2)** pages.
3. Answer any **(4)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary
6. Involvement in any activity that is considered as an exam offense will lead to punishment
7. Use blue or black ink to answer the questions.
8. Clearly state your index number in your answer script

1. (a) State and sketch the domain of the function $f(x, y) = \sqrt{x} + \sqrt{y} + \sqrt{x^2 + y^2 - 4}$.

(b) Sketch the level curves of the function $f(x, y) = \sqrt{5x^2 + y^2} - 2x$ when it takes the values 1 and 2.

(c) Find the following limits if they exist

(i) $\lim_{(x, y)\to(0, 0)} \dfrac{x^2 y^2}{x^4 + y^4}$, (ii) $\lim_{(x, y)\to(1, 1)} \dfrac{xy - y - 2x + 2}{x - 1}$.

justifying your answer.

(d) Discuss the continuity of the following function at (1, 1).

$$f(x, y) = \begin{cases} \dfrac{xy - y - 2x + 2}{x - 1} & \text{if } (x, y) \neq (1, 1) \\ -1 & \text{if } (x, y) = (1, 1). \end{cases}$$

(You may use your conclusion regarding c(ii).)

2. (a) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D. Explain briefly how you could determine its nature.

(b) Find the maximum and minimum values of the function $f(x, y) = x^3 + y^3 - 3x^2 + 3y^2$ and determine their nature.

(c) Prove that the vector field $\underline{F} = (2x + yz)\underline{i} + (2y + zx)\underline{j} + (2z + xy)\underline{k}$ is conservative. Find the corresponding scalar potential function ϕ such that $\underline{F} = \nabla\phi$.

3. (a) Prove that grad ϕ is a vector normal to the contour surface $\phi(x, y, z) = c$, where c is a constant.

(b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is given by $(x - x_0)\left(\dfrac{\partial F}{\partial x}\right)_P + (y - y_0)\left(\dfrac{\partial F}{\partial y}\right)_P + (z - z_0)\left(\dfrac{\partial F}{\partial z}\right)_P = 0$.

(ii) Using the above result, find the equation of the tangent plane to the surface $z = 14 - x^2 - y^2$ at the point $P(1, 2, 9)$.

(c) Suppose that a mountain has the shape of an elliptic paraboloid $z = c - ax^2 - by^2$, where a, b and c are positive constants, x and y are the east-west and north-south map coordinates, and z is the altitude above sea level. At the point (1, 2), in what in what direction is the altitude increasing most rapidly? If a marble were released at (1, 2), in what direction would it begin to roll?

4. (a) State Gauss' Divergence Theorem.

(b) Verify the above theorem considering the vector field $\underline{F} = ax\underline{i} + by\underline{j} + cz\underline{k}$ where a, b and c are constants, defined over the spherical region S bounded by $x^2 + y^2 + z^2 = 1$.

5. (a) State Stokes' Theorem.

(b)Verify Stokes' Theorem considering the vector field $\underline{F} = (2x - y)\underline{i} - yz^2\underline{j} - y^2z\underline{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = 1;\ z \geq 0$ and C is its boundary.

(c) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ be the position vector of the point (x, y, z) and $r = |\underline{r}|$. Show that (for $r \neq 0$)

(i) $\underline{\nabla}\left(\frac{1}{r^2}\right) = -\frac{2\underline{r}}{r^4}$,   2(ii) $\underline{\nabla}.\left(\frac{\underline{r}}{r^2}\right) = \frac{1}{r^2}$,

where $\underline{\nabla}$ has a standard meaning.

6. (a) Suppose that S is a plane surface lying in the xy–plane and bounded by a closed curve C. If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$.

(b) Verify the above result for the integral $\oint_C \left[(xy + y^2)\,dx + x^2dy\right]$, where C is the closed curve of the region bounded by the curves $y = x$ and $y = x^2$.