00119

The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	: Mathematics
Level	: 05
Name of the Examination	: Final Examination
Course Code and Title	: ADU5302- Mathematical Methods
Academic Year	: 2020/2021
Date	: 02.12.2021
Time	: 9.30 a.m.-11.30 a.m.
Duration	: 2 Hours

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of 06 questions in 03 pages.
3. Answer any 04 questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary
5. Relevant log tables are provided where necessary.
6. Having any unauthorized documents/ mobile phones in your possession is a punishable offense
7. Use blue or black ink to answer the questions.
8. Circle the number of the questions you answered in the front cover of your answer script.
9. Clearly state your index number in your answer script

1. (a) Find the inverse Laplace transform of each of the following:
(where a standard notation has been used.)

(i) $\dfrac{s}{(s-2)^2(s+1)}$

(ii) $\dfrac{3s-137}{s^2+2s+401}$

(b) Using the convolution theorem, find the inverse Laplace transform of

$$H(s)=\frac{1}{(s+1)(s^2+1)}.$$

(c) Solve the following boundary value problem using the Laplace transform method:

$$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x=50\sin t \quad ;x=0 \text{ and } \frac{dx}{dt}=0 \text{ when } t=0.$$

2. Consider the boundary value problem

$$\frac{d^2y}{dx^2}+\mu y=0 \;\; ;\mu\in\Re$$
$$y(-\pi)=y(\pi)$$
$$y'(-\pi)=y'(\pi)$$

(a) Find the eigenvalues and eigenfunctions of the problem.

(b) Obtain a set of functions, which are orthonormal in the interval $-\pi\le x\le\pi$.

3. (a) Find the Fourier Series of $f(x)=\sqrt{1-\cos x}$ in the interval $-\pi<x<\pi$.

(b) Find the Fourier sine series and the Fourier cosine series of the following function:

$$f(x)=\begin{cases}\dfrac{2k}{L}x & \text{if } 0<x<\dfrac{L}{2}\\[2mm] \dfrac{2k}{L}(L-x) & \text{if } \dfrac{L}{2}<x<L\end{cases}$$

4. (a) The Gamma function denoted by $\Gamma(p)$ corresponding to the parameter p is defined by the improper integral $\Gamma(p)=\int_0^\infty e^{-t}t^{p-1}dt, \quad (p>0)$.

Evaluate each of the following:

(i) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$

(ii) $\int_0^\infty x^m e^{-ax^n}\, dx$; where m, n, a are positive constants.

(b) The Beta function denoted by $\beta(p,q)$ is defined by

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx, \quad \text{where } p \text{ and } q \text{ are positive parameters.}$$

Use Gamma function and Beta function to evaluate each of the following integrals:

(i) $\int_0^{1/2} x^3(1-4x^2)^{1/2}\, dx$

(ii) $I = \int_0^{2a} x^2\sqrt{2ax-x^2}\, dx$; a is a positive constant.

5. Let $J_p(x)$ be the Bessel function of order p given by the expansion

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p}.m!\Gamma(p+m+1)}$$

Prove each of the following results:

(a) $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$

(b) $\int J_3(x)dx + J_2(x) + \frac{2}{x}J_1(x) = 0$

(c) $J_0''(x) = \frac{1}{2}[J_2(x) - J_0(x)]$

(Hint: You may use the following recurrence relations, if necessary, without proof.)

$$\frac{d}{dx}\{x^p J_p(x)\} = x^p J_{p-1}(x)$$

$$\frac{d}{dx}\{x^{-p} J_p(x)\} = -x^{-p} J_{p+1}(x).$$

$$J_p'(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$$

$$J_p'(x) = \frac{1}{2}\{J_{p-1}(x) - J_{p+1}(x)\}$$

6. The Rodrigue's formula for the n^{th} degree Legendre polynomial denoted by $Pn(x)$ is given as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

$P_n(x)$ is also given by the sum

$$P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!}.x^{n-2m} \quad , \quad n = 0,\ 1,\ 2, \dots ,$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$, whichever is an integer.

(a) Write down the function $5x^3 - 3x^2 + 5x - 1$ in terms of Legendre Polynomials.

(b) Prove each of the following results:

(i) $\int_{-1}^{1} p_n(x)dx = 0$, for $n \neq 0$

(ii) $\int_{-1}^{1} p_n(x)(1-2xt+t^2)^{-\frac{1}{2}}dx = \frac{2t^n}{2n+1}$

where n is a positive integer.

(Hint: $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n.p_n(x)$)