THE OPEN UNIVERSITY OF SRI LANKA
Faculty of Engineering Technology
Department of Mathematics & Philosophy of Engineering

Bachelor of Technology Honors in Engineering

Final Examination (2020/2021)
MHZ3531 Engineering Mathematics IA

Date: 19th January 2022 (Wednesday) **Time: 1400 hrs. – 1700 hrs.**

Instructions:

- *Answer* ***five (05)*** *questions only.*
- *Number of pages in the paper is five (05).*
- *All the symbols are in standard notation unless they are defined.*

Q1

(a) Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$. Prove that the roots of the equation $|A - \mu I| = 0$ are 1 and 5, where I and 0 are the unit matrix and the zero matrix of order two respectively. (15%)

(i) Taking $\mu_1 = 1$ and $\mu_2 = 5$, if $AX_1 = \mu_1 X_1$ and $AX_2 = \mu_2 X_2$, Prove that $X_1 = s\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $X_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,where s and t are parameters. (15%)

(ii) Prove that $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent. (10%)

(b) Let $P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$.

(i) Show that $A = PDP^{-1}$. (15%)

(ii) Hence find A^6. (15%)

(c) Using row and column operations, Prove that

$$\begin{vmatrix} 1 + a^2 - b^2 & 2ab & -2b \\ 2ab & 1 - a^2 + b^2 & 2a \\ 2b & -2a & 1 - a^2 - b^2 \end{vmatrix} = (1 + a^2 + b^2)^3. \quad (30\%)$$

Q2

(a) Let $f(x) = \dfrac{x^2 + x + 2}{x - 1}$.

(i) Find the vertical and slant asymptotes of the curve of $y = f(x)$. (20%)

(ii) Find $\dfrac{dy}{dx}$ and determine the sign of $\dfrac{dy}{dx}$ as x varies from $-\infty$ to ∞. (15%)

(iii) Find the coordinates of the turning points of the curve of $y = f(x)$. (10%)

(iv) Sketch the graph of $y = f(x)$. (25%)

(b) If $z = x^3 - 3x^2y - 3xy^2 + y^3$. Prove the followings.

(i) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 3z$ (15%)

(ii) $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = 6z.$ (15%)

Q3

(a) State De Moivre's theorem. (10%)

(i) By using above theorem, express $\cos 6\theta$ and $\sin 6\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. (20%)

(ii) Hence prove that, $\tan 6\theta = \dfrac{2\tan\theta(3 - 10\tan^2\theta + 3\tan^4\theta)}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$. (10%)

(b) If $z = x + iy$, determine the loci in the Argand diagram, defined by

(i) $Re(z) = |z - 2|$. (15%)

(ii) $|z - 1| = \sqrt{2}|z - i|$. (15%)

(c) Prove that $2 + 3i$ is a root of $2z^4 - 11z^3 + 39z^2 - 43z + 13 = 0$. Find the other roots of the equation. (30%)

Q4

(a) Find the equation of the normal to the hyperbola $xy = p^2$ at the point $P \equiv \left(pt, \frac{p}{t}\right)$. (15%)

(i) This normal meets the curve at the point Q. Find the coordinates of Q. (20%)

(ii) Show that the locus of the mid-point of PQ is $p^2(x^2 - y^2) + 4x^3y^3 = 0$. (15%)

(b) Prove that the straight line $\frac{x-4}{3} = \frac{y-1}{2} = \frac{z-3}{1}$ intersects a line of intersection of the planes $x + y + 2z = 4$ and $3x - 2y - z = 3$. Find the equation of the plane that contains these two lines. (50%)

Q5

(a) Using an integrating factor, find the general solution of the differential equation
$$x\frac{dy}{dx} + x + 2y = 0.$$ (35%)

(b) Using the substitution $y = vx$, Find the solution of the differential equation
$$(y - x)\frac{dy}{dx} + (2x + 3y) = 0.$$ (35%)

(c) Solve the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$, subject to the boundary conditions $y(0) = 0$ and $y'(0) = 1$. (30%)

Q6

(a) Define a dot product of two vectors. (10%)

(b) The position vectors of two points A and B with respect to an origin O are $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Let C be the point such that $\overrightarrow{OC} = \frac{\lambda}{3}\overrightarrow{OB}$ and let D be the point such that $\overrightarrow{OD} = \frac{\lambda}{2}\overrightarrow{AB}$. [Hint: A, B and O are not collinear].

(i) Express $\overrightarrow{AC}$ and $\overrightarrow{AD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. (20%)

(ii) If $\overrightarrow{AD} = \frac{3}{2}\overrightarrow{AC}$, then find the value of λ. (20%)

(c) Let p and q be any real numbers and $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors. Show that for any scalar μ, the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are given by,

$$\boldsymbol{x} = \frac{(1-\mu q)}{p}\boldsymbol{u} - \frac{q}{|\boldsymbol{u}|^2}(\boldsymbol{u} \times \boldsymbol{v})$$

$$\boldsymbol{y} = \mu \boldsymbol{u} + \frac{p}{|\boldsymbol{u}|^2}(\boldsymbol{u} \times \boldsymbol{v})$$

satisfy the equations, $p\boldsymbol{x} + q\boldsymbol{y} = \boldsymbol{u}$ and $\boldsymbol{y} \times \boldsymbol{u} = p\boldsymbol{v}$. (30%)

(d) Show that a plane with equation $x + 2y + 3z = 1$ has $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ as a normal.

(20%)

Q7

(a) Show that the positive root of the square root of 2 lies between 1 and 2.

(10%)

(b) An iterative formula for finding a square root of any positive real number a is given below.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(i) Show that the above iterative formula is consistent for finding a square root of any positive real number a. (10%)

(ii) Show that the above iterative formula is convergent to find the square root of 2. (10%)

(iii) If the above iterative formula is convergent, then find the square root of 2, correct to two decimal places. (20%)

(iv) By considering the method of Bisection and the iterative formula method, which method is efficient? Explain your answer. (25%)

(c) Let consider the following values of the function $y = f(x)$ at some discrete values are given in the following table.

x	0	1	2	3	4
y	1	5	8	10	15

Using the Newton's interpolation method, find the value of $y(1.3)$. (25%)

Q8

(a) A box contains $t_1 + t_2 + t_3$ tags, where t_1 tags are numbered 1 and t_2 tags are numbered 2 and t_3 tags are numbered 3. There are 3 urns and they are numbered 1,2 and 3. Urn number i contains w_i white balls and b_i black balls ($i = 1,2,3$). A tag is selected at random from the box and then a ball is drawn at random from the urn of the same number as the tag selected. Then find the probability that the ball selected is from the urn numbered 2, given that the ball is white. (50%)

(b) The random variable X has probability density function $f(x)$ given by

$$f(x) = \begin{cases} kx(a-x), & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Where k and a are positive constants and $a \ge 4$.

(i) Show that $k = \dfrac{3}{8(3a-8)}$. (20%)

(ii) Given that $E(X) = 2$. Show that $a = 4$ and write down the value of k. (10%)

(iii) Calculate the variance and median of X. (20%)

End