00065

THE OPEN UNIVERSITY OF SRI LANKA
Faculty of Engineering Technology
Department of Mathematics & Philosophy of Engineering

Bachelor of Technology Honors in Engineering / Bachelor of Software Engineering Honors

Final Examination (2020/2021)
MHZ5355: Discrete Mathematics

Date: 30th January 2022 (Sunday) **Time: 14:00 – 17:00**

Instruction:

- Answer only five questions.
- Please answer a total of five questions choosing at least one from each single section.
- All symbols are in standard notation and state any assumption that you made.

SECTION – A

Q1.

i. Prove the following statements using Mathematical induction.

a) $7^n - 2^n$ is divisible by 5 for all $n \geq 1$; [15%]

b) $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ for all $n \geq 1$. [15%]

ii. Let $a, b,$ and c be any integer numbers. Prove that,

a) if $a|b$ and $a|c$, then $a|(3b - 7c)$, [10%]

b) if $a|b, a > 0$, and $b > 0$ then $a \leq b$, [10%]

c) if $a|b$ and $b|c$, then $a|c$. [10%]

iii.

a) Find the $gcd(285,741)$, and find the integers x, y such that $\gcd(285, 741) = 285x + 741y$ by using the Euclidean Algorithm. [15%]

b) Determine all integer solutions of the following Diophantine equation:

$$285x_0 + 741y_0 = 855.$$ [10%]

c) Find the least common multiple (lcm) of 285 and 741. [15%]

Q2.

i. Let $gcd(a,b) = 1$. Show that $gcd(2a+b, a+2b) = 1$ or 3 **[25%]**

ii. Let a, b, c and d be integers. Let n be a positive integer. Show that

a) if $a \equiv b(mod\ n)$ and $c \equiv d(mod\ n)$, then $a + c \equiv (b+d)(mod\ n)$ and $ac \equiv bd(mod\ n)$. **[20%]**

b) if $a \equiv b(mod\ n)$, then $a^k \equiv b^k(mod\ n)$ for any $k \in \mathbb{Z}^+$. **[15%]**

iii. Solve the following system of congruence: **[40%]**

$$13x \equiv 4(mod\ 7)$$
$$x \equiv 7(mod\ 12)$$
$$x \equiv 4(mod\ 17).$$

SECTION – B

Q3.

i. Determine whether the operations " $*$ " , $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ defined below are binary operation or not. Prove your answer if " $*$ " is a binary operation, explain the reason if " $*$ " not a binary operation. **[30%]**

a) $x * y = \frac{x}{y}$; $\quad x, y \in \mathbb{N}$,

b) $x * y = |x - y|$; $\quad x, y \in \mathbb{N}$,

c) $x * y = x + y - xy$; $\quad x, y \in \mathbb{N}$.

ii. Let A be any nonempty set with the operation " $*$ " defined by $a * b = a^2 + b^2$.

a) Is this operation associative?

b) Is this operation commutative?

Justify your answer. **[25%]**

iii. Define an abelian group $(G, *)$ in usual notation.

Let $G = \{(a,b): a, b \in \mathbb{R}; a \neq 0\} = \mathbb{R} \setminus \{0\} \times \mathbb{R}$. Let a binary operation " $*$ " defined by $(a,b) * (c,d) = (ac, b+d)$ for all $(a,b), (c,d) \in G$. **[45%]**

a) Show that $(G,*)$ is a group.

b) Is the group $(G,*)$ Abelian? Justify your answer.

Q4.

i. Define a semi-group $(G, \#)$ in usual notation.
Let operations "#" : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, defined as follows.

a) "$a \# b = ab + 5$ for all $a, b \in \mathbb{R}$;
b) "$c\# d = \frac{2}{3} cd$" for all $c, d \in \mathbb{R}$;
c) "$x \# y = 2(x + y) + 3$" for all $x, y \in \mathbb{R}$.

Verify that where $(\mathbb{R}, \#)$ a semi-group or not for each of the above cases. [45%]

ii. Let $G = \{1, -1\}$. Show that $(G, *)$ is a group, where "$*$" is the ordinary multiplication. [25%]

iii. Define a homomorphism for group in usual notation. [30%]
Let $T = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}\right\}$ be a group under matrix addition.
Let $f: T \rightarrow \mathbb{Z}$ be defined by $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$.
Prove that f is a homomorphism of T on to $\mathbb{Z}$, where $\mathbb{Z}$ is a group under addition.

SECTION – C

Q5.

i. Determine which of the following are simple, by drawing each of them.

a) $G_1 = \{V_1, E_1\}$ where $V_1 = \{1, 2, 3, 4, 5, 6\}$ and
$E_1 = \{\{x, y\}, 3x + y \text{ is even and } x < y\}$ [10%]

b) $G_2 = \{V_2, E_2\}$ where $V_2 = \{1, 2, 3, 4, 5, 6, 7\}$ and
$E_2 = \{ \{i, j\}: |i - j| \leq 3; \text{ and } i \leq j\}$ [10%]

c) $G_3 = \{V_3, E_3\}$ where $V_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and
$E_3 = \{\{m, n\}: m \times n \text{ is a multiple of } 10 \text{ and } m < n\}$ [10%]

ii. Let G be a graph of 10 vertices and 15 edges such that every vertex is of degree 2 or 4. How many vertices of G degree? Construct one such graph G.
[25%]

iii. G is the graph whose adjacency matrix M is given by

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

a) Without drawing a graph of G, explain whether G is connected or not. **[15%]**

b) If $V(G) = \{p, q, r, s\}$ then find the number of paths of length four joining vertices r and s. **[10%]**

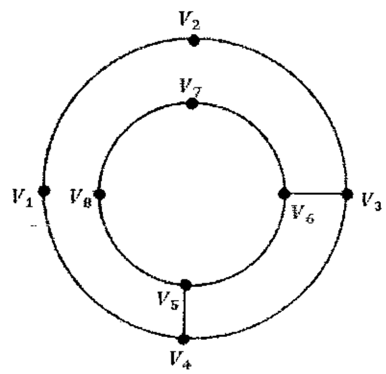
c) Draw the graph of a adjacency matrix M. **[05%]**

iv. Find the number of vertices n such that the complete graph has at least 1200 edges. **[15%]**

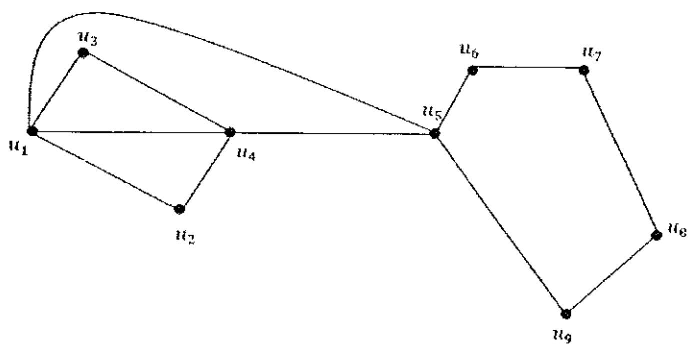
Q6.

i. Determine whether following graph are Eulerian or not. Which of them are Hamiltonian ? Justify your answers. **[30%]**

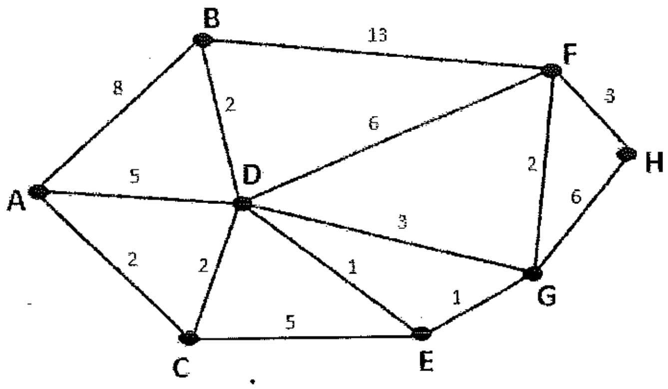
a)



b)



ii. Use Dijkstra's algorithm to find the shortest route between node A and every other node in the following network. **[60%]**



iii. Draw a tree with 16 vertices at least 8 vertices have degree one. **[10%]**

SECTION – D

Q7.

i. Iterate the Eco-system growth model relationship $t_{n+1} = \lambda t_n - \lambda t_n^2$, where $\lambda = 2.3$ and $t_0 = 0.4$ (at least 7 iteration steps are required) and draw the graph for the relation. Hence deduce t_n as $n \to \infty$.
(Up to 4 decimal places probably) **[20%]**

ii. Consider the iterations given by $Z_{n+1} = Z_n^2$ and suppose that $Z_0 = \frac{1}{\sqrt{2}}(1 + i)$.

a) Find Z_1, Z_2, Z_3 and Z_4.
b) Plot them in a same Argon diagram
c) Discuss the behavior of Z_n as $n \to \infty$.

[20%]

iii. Suppose a system with two unknowns x and y, is modeled in the form of a system of differential equations

$$\frac{dx}{dt} = 2x - 6y$$
$$\frac{dy}{dt} = x + 7y$$

with initial conditions $x = 1$ and $y = 0$ when $t = 0$.
Find two iterations for the unknowns x and y at t. **[60%]**

Q8.

i. Let $L = \{001, 10, 111\}$ and $M = \{\varepsilon, 01, 001\}$ be languages. Find the concatenations LM and ML. **[20%]**

ii. Show that the string $a - b * c + d$ is a sentence generated by the grammar G, where $G = \{\{S, E, T, F\}, \{+, *, -, /, a, b, c, d\}, P, S\}$ and starting symbol S and production P.
P is the set of rules defined as follows: **[25%]**

$S \to E,$	$T \to F$
$E \to E + T,$	$F \to (E)$
$E \to E - T,$	$F \to a,$
$E \to T,$	$F \to b,$
$T \to T * F$	$F \to c,$
$T \to T/F$	$F \to d$

iii. Let $M = \{S, I. \delta, S_0, F\}$ be a Non-Deterministic Finite Automata (NDFA).
Where S is a finite set of state, I is a finite set of input symbols, δ is the transition function, S_0 is the initial state, and F is the set of final states.
Transition Table for the above Non-Deterministic Finite Automata as follows:

States	Inputs	
	a	**b**
0	{0, 1}	{0, 4}
1	{1}	{2}
2	{2, 3}	-
3	{3}	{3}
4	{5}	-
5	{5}	{4, 5}

The initial state is **0**, and the set of final states is $\{\mathbf{3}, \mathbf{5}\}$.

a) Depict the finite automaton's transition graph. **[15%]**

b) Show that the string ***ababab*** is accepted by the Non-deterministic Finite Automaton by applying the transition function. **[40%]**

--- END ---