The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	: Mathematics
Level	: 05
Name of the Examination	: Final Examination
Course Code and Title	: ADU5302- Mathematical Methods
Academic Year	: 2021/2022
Date	: 03.11.2022
Time	: 9.30 a.m.-11.30 a.m.
Duration	: 2 Hours

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of 06 questions in 03 pages.
3. Answer any 04 questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary.
5. Relevant log tables are provided where necessary.
6. Having any unauthorized documents/ mobile phones in your possession is a punishable offense.
7. Use blue or black ink to answer the questions.
8. Circle the number of the questions you answered in the front cover of your answer script.
9. Clearly state your index number in your answer script.

1.(a) Find the Laplace transform $L(t)$ of $\cos t \cos 2t$.

(b) Find the inverse Laplace transform $L^{-1}\left\{\ln\left(1+\frac{\omega^2}{s^2}\right)\right\}$.

(c) Using the convolution theorem, find the inverse Laplace transform of

$$H(s)=\frac{s}{(s+2)(s^2+9)}.$$

(d) Solve the following boundary value problem using the Laplace transform method:

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=4 \quad \text{subject to} \quad y(0)=2,\ y'(0)=3.$$

2. Consider the boundary value problem,

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right)+\lambda\left(\frac{1}{x}\right)\phi=0,\ \ 1<x<2;$$

$$\phi(1)=\phi(2)=0.$$

(a) Show that this is a Sturm-Liouville problem.

(b) Show that, when λ is non- negative

$$\phi(x)=A\cos(\sqrt{\lambda}\ln x)+B\sin(\sqrt{\lambda}\ln x)$$

is the general solution of the given differential equation.

(c) Find the eigenvalues and eigen functions of the problem.

3. (a) Determine the Fourier series for the function given below:

$$f(x)=\begin{cases}\frac{2x}{\pi} & 0<x\le\pi\\ 2 & \pi\le x\le 2\pi.\end{cases}$$

and $0\le x\le 2\pi$.

(b) Find the Fourier sine series and the Fourier cosine series of the following function:

$f(x)=2x-1;\ 0<x\le 1$

4. (a) The Gamma function denoted by $\Gamma(p)$ corresponding to the parameter p is

defined by the improper integral $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt, \quad (p > 0).$

Evaluate each of the following using Gamma functions:

(i) $\int_0^1 (x \log x)^3 dx$

(ii) $\int_0^\infty \frac{x^a}{a^x} dx$

(b) The Beta function denoted by $\beta(p,q)$ is defined by

$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx,$ where p and q are positive parameters.

Use Gamma function and Beta function to evaluate each of the following integrals:

(i) $\int_0^{2\pi} \sin^8 \theta d\theta$

(ii) $\int_0^1 x^4 \sqrt{1-x^2} dx.$

5. Let $J_p(x)$ be the Bessel function of order p given by the expansion

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p}.m!\Gamma(p+m+1)}$$

(a) Prove each of the following results:

(i) $J_{-n}(x) = (-1)^n J_n(x)$ for $n = 1, 2, 3, \ldots$

(ii) $J_n''' = \frac{1}{8}[J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}]$.; where $'''$ denotes a standard notation.

(b) Find $J_0(x)$ and $J_1(x)$.

(c) Show that $J_n(x)$ is an even function when n is even and is an odd function when n is odd.

(Hint: You may use the following recurrence relations, if necessary, without proof.)

$\frac{d}{dx}\{x^p J_p(x)\} = x^p J_{p-1}(x)$

$$\frac{d}{dx}\left\{x^{-p}J_p(x)\right\} = -x^{-p}J_{p+1}(x).$$

$$J_p'(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)$$

$$J_p'(x) = \frac{1}{2}\left\{J_{p-1}(x) - J_{p+1}(x)\right\}$$

6. The Rodrigue's formula for the n^{th} degree Legendre polynomial denoted by $Pn(x)$ is given as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

$P_n(x)$ is also given by the sum

$$P_n(x) = \sum_{m=0}^{M}\frac{(-1)^m(2n-2m)!}{2^n m!(n-m)!(n-2m)!}.x^{n-2m}\ , \quad n = 0,\ 1,\ 2,\ldots,$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$, whichever is an integer.

(a) Prove that $\left(1-x^2\right)P_{n-1}' = n\left(xP_{n-1} - P_n\right)$

(Hint: You may use the following recurrence relations, if necessary, without proof.)

$$P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$$
$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x).$$
$$xP_n'(x) = nP_n(x) + P_{n-1}'(x).$$

(b) Express x^3 and x^4 in terms of Legendre Polynomials.

(c) Express the polynomial $f(x)$ in terms of Legendre Polynomials where

(i) $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$.

(ii) $f(x) = 2x + 10x^3$