The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc. / B. Ed. Degree Programme

Department	**: Mathematics**
Level	**: 05**
Name of the Examination	**: Final Examination**
Course Title and - Course Code	**: Combinatorics–PEU5302**
Academic Year	**: 2021/22**
Date	**: 19.10.2022**
Time	**: 1.30 p.m. To 3.30 p.m.**
Duration	**: Two Hours.**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **(6)** questions in **(4)** pages.
3. Answer any **(4)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully and clearly labelled diagrams where necessary
6. Involvement in any activity that is considered as an exam offense will lead to punishment.
7. Use blue or black ink to answer the questions.
8. Clearly state your index number in your answer script

(01) (a) In how many ways can 5 boys and 4 girls be arranged on a bench if

(i) there are no restrictions?

(ii) boys and girls alternate?

(iii) boys and girls are in separate groups?

(iv) Anne and Jim wish to stay together?

(b) At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

(i) there are no restrictions.

(ii) men and women alternate.

(iii) Ted and Carol must sit together.

(iv) Bob, Ted, and Carol must sit together.

(v) Neither Bob nor Carol can sit next to Ted.

(c) If 4 Mathematics books are selected from 6 different Mathematics books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

(i) there are no restrictions?

(ii) the 4 Mathematics books remain together?

(iii) a Mathematics book is at the beginning on a shelf?

(iv) Mathematics and English books are alternate.

(v) if a Mathematics book is at the beginning and an English book is in the middle of the shelf.

(02) (a) Let A and B be any two subsets of the Universal set $\cup$.

Prove that $(A \cup B)^c = A^c \cap B^c$. Hence, show that

$\forall n \in \mathbb{N}, \left(\bigcup_{k=1}^{n} A_k \right)^c = \bigcap_{k=1}^{n} \left(A_k^c \right)$ where A^c is the complement of A.

(b) Draw a flow chart to check whether a given number is a prime or not.

(c) (i) Prove that, if 7 numbers are selected from numbers 1 through 11, then there will always be a pair whose sum is 12.

(ii) Find the minimum number of students that should be in a class to guarantee that at least 3 have born in the same month.

(d) (i) How many different 8-letter words are possible using the letters of the word "SYLLABUS"?

(ii) If a word from (i) is chosen at random, find the probability that the word contains the two Ss together.

(03) (a) Let $C_1, C_2, \ldots, C_M$ be a partition of the sample space S, A and B be two events. Suppose that A and B are conditionally independent.

It is given that the event B is independent of all C_i for all $i \in \{1, 2, \ldots, M\}$. Prove that A and B are independent.

(b) A fair coin tossed three times.

(i) What is the probability of getting three Heads?

(ii) What is the probability of getting exactly one Head?

(iii) If it is given that the first toss is Head, then find the probability of getting at least two Heads?

(c) In a school there are 148 students in Years 12 and 13 studying Science, Humanities, or Arts subjects. Of these students, 89 wear glasses and others do not. There are 30 Science students of whom 18 wear glasses. The corresponding figures for the Humanities students are 68 and 44 respectively.

A student is chosen at random. Find the probability that this student;

(i) is studying Arts subjects,

(ii) does not wear glasses, given that the student is studying Arts subjects.

Amongst the Science students, 80% are right-handed and the corresponding percentages for Humanities and Arts students are 75% and 70% respectively. A student is again chosen at random.

(iii) Find the probability that this student is right-handed.

(iv) Given that this student is right-handed, find the probability that the student is studying Science subjects.

(04) (a) Using Mathematical induction, prove Binomial theorem.

(b) Using Mathematical induction, prove that the given statement is true for all positive integers n.

(i) $3^n > n^2$ for all positive integers n greater than 2.

(ii) $2+4+6+...+2n = n(n+1)$.

(iii) $n^2 - 3n + 4$ is even.

(05)(a) Let n and r be two positive integers such that $1 \le r \le n$. Use a combinatorial argument to prove the Pascal's identity ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$. Hence, deduce that

$${}^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{n+r}C_r = {}^{n+r+1}C_r.$$

(b) Prove that $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + ... + C_{n-r}C_n = \dfrac{(2n)!}{(n-r)!(n+r)!}$.

(c) Let n, m and r be three positive integers such that $1 \le r \le n$ and $1 \le r \le m$.

Prove each of the following statements using ONLY combinatorial arguments:

(i) $n \times {}^{n-1}C_{r-1} = r \times {}^{n}C_r$.

(ii) ${}^{n}C_2 + {}^{m}C_2 + nm = {}^{n+m}C_2$.

(iii) ${}^{3n}C_2 - {}^{2n}C_2 - {}^{n}C_2 = 2n^2$.

(06). (a) In the multinomial expansion of $(x+y+z+w)^{100}$, find the number of terms of the form $x^p y^q z^r w^s$ where p, q, r and s are non-negative integers such that $p+q+r+s=100$ and $p \geq 95$.

(b) Find the multinomial coefficient of each of the following terms with respect to the expansion of the given expression.

(i) $x^{99} y^{60} z^{14}$ in the expansion of $\left(2x^3+y-z^2\right)^{100}$.

(ii) x^5 in the expansion of $\left(p-qx+rx^2\right)^6\left(p+qx-rx^2\right)^6$.

(iii) $a^2 b^3 c^2 d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$.

(iv) $x^3 y^2 z^4 w^2$ in the expansion of $(x+y+z+w)^{11}$.

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