The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2021/2022
Pure Mathematics - Level 05
PEU5305 – Complex Analysis I
Duration: - Two hours

Date: - 25-10-2022 **Time: - 01.30 p.m. – 3.30 p.m.**

Answer **FOUR** Questions **ONLY**.

1.

a) Prove that for every non-zero complex number z, there is a complex number which we shall denote by z^{-1} such that $z \cdot z^{-1} = 1$.

b) Let $z_1, z_2 \in \mathbb{C}$. Show that $z_1 \overline{z_2} + \overline{z_1} z_2$ is real.

c) Let $z \in \mathbb{C}$ and $\operatorname{Im} z \neq 0$. Show that $\left(\frac{z+1}{z-1}\right)^2$ is real if and only if $z \cdot \overline{z} = 1$.

d) Prove that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$ for all $z \in \mathbb{C}$.

2.

a) Give the definition of each of the following:

i. A bounded subset E of $\mathbb{C}$.

ii. An open subset E of $\mathbb{C}$.

b) Sketch the following sets and determine whether each is open, each is closed, each is bounded, and each is a region:

i. $E_1 = \{z \in \mathbb{C} : \operatorname{Re}(z) = \operatorname{Im}(z)\}$.

ii. $E_2 = \{z \in \mathbb{C} : |z-2| \leq 5\}$.

iii. $E_3 = \left\{z \in \mathbb{C} : z = r(\cos\theta + i\sin\theta),\ r > 0,\ \frac{\pi}{4} < \theta < \frac{\pi}{2}\right\}$.

c) Prove or disprove each of the following:

i. The set of all interior points of the set $S = \{1 + in : n = 1, 2, 3, \ldots\}$ is S.

ii. An arbitrary union of open sets is open.

3.

a) Give the definition of each of the following:

i. A complex - valued function $f(z)$ is analytic in an open subset of $\mathbb{C}$.

ii. A complex - valued function $f(z)$ is analytic at a point $z_0 \in \mathbb{C}$.

b) Determine where the function $f(z) = 2x^3 + xy^2 + i\left(\frac{y^3}{3} + 6x^2 y\right)$ is differentiable and where it is analytic.

c) Prove that the function $u(x, y) = 2x(1 - y)$ is harmonic. Find a function $v(x, y)$ such that $f(z) = u + iv$ is analytic in $\mathbb{C}$.

d) Let $f(z) = u + iv$ be analytic in a region G. Show that if $\operatorname{Re} f(z)$ is constant in G, then $f(z)$ is constant in G.

4.

a) Let $f(z) = \dfrac{1}{(z-1)(z-2)}$. Find the Laurent series expansion of $f(z)$ in each of the following annuli:

i. $1 < |z| < 2$,

ii. $|z| > 2$,

iii. $0 < |z - 1| < 1$.

b) Show that the function $f(z) = \dfrac{e^z}{z^3}$ has a pole of order 3 at $z = 0$.

c) Find and classify the singularities of the function $f(z) = \dfrac{e^z}{(z-1)(z+i)^2}$.

5.

a) State Cauchy's Integral Formula.

b) Using Cauchy's Integral Formula, evaluate the integral $\int_C \frac{\cos z}{z^2-6z+5}\,dz$, where C is the circle with radius 4 centered at 0, oriented counterclockwise.

c) Apply Cauchy's integral formula to $\cos z$ with the contour $z=e^{i\theta};\ 0\le\theta\le 2\pi$ to show that $\int_0^{2\pi}\cos(\cos\theta)\cosh(\sin\theta)\,d\theta=2\pi$ and $\int_0^{2\pi}\sin(\cos\theta)\sinh(\sin\theta)\,d\theta=0$.

6.

a) State Cauchy's Residue Theorem. Use the residue theorem to calculate each of the following integrals:

i. $\int_C \frac{5z}{z^2+4}\,dz$, where C is the circle $|z|=2$, oriented counterclockwise.

ii. $\int_C \cot z\,dz$, where C is the circle $|z|=4$, oriented counterclockwise.

b) Use the residue theorem to show that $\int_0^{2\pi}\frac{4}{5+4\cos\theta}\,d\theta=\frac{8\pi}{3}$.