The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc. / B. Ed Degree Programme

Department	**: Mathematics**
Level	**: 04**
Name of the Examination	**: Final Examination**
Course Title and - Code	**: Vector Calculus – ADU4302**
Academic Year	**: 2021/22**
Date	**: 18.10.2022**
Time	**: 1.30 p.m. To 3.30 p.m.**
Duration	**: Two Hours.**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **(6)** questions in **(2)** pages.
3. Answer any **(4)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Involvement in any activity that is considered as an exam offense will lead to punishment
6. Use blue or black ink to answer the questions.
7. Clearly state your index number in your answer script.

1. (a) State and sketch the domain of the function $f(x,y) = \sqrt{x^2 + y^2 - 4}$.

(b) Sketch at least three level curves of the function $f(x,y) = \sqrt{x^2 + y^2 - 4}$.

(c) Find the following limits if they exist

(i) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$, (ii) $\lim\limits_{(x,y)\to(0,0)} \frac{x^4-4y^4}{x^2+2y^2}$.

justifying your answer.

(d) Discuss the continuity of the following function at (0, 0).

$$f(x,y) = \begin{cases} \frac{x^4-4y^4}{x^2+2y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

(You may use your conclusion regarding c(ii).)

2. (a) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D. Explain briefly how you could determine its nature.

(b) Find the maximum and minimum values of the function $\boldsymbol{f(x,y) = x^4 + y^4 - 16xy}$ and determine their nature.

(c) Prove that the vector field $\underline{F} = 2xy\underline{i} + (x^2 + 2yz^3)\underline{j} + (3y^2z^2 + 2z)\underline{k}$ is conservative. Find the corresponding scalar potential function ϕ such that $\underline{F} = \underline{\nabla}\phi$.

3. (a) Prove that grad ϕ is a vector normal to the contour surface $\phi(x,y,z) = c$, where c is a constant.

(b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is given by $(x-x_0)\left(\frac{\partial F}{\partial x}\right)_P + (y-y_0)\left(\frac{\partial F}{\partial y}\right)_P + (z-z_0)\left(\frac{\partial F}{\partial z}\right)_P = 0$.

(ii) Using the above result, show that the equation of the tangent plane to the surface $xy + yz + zx = 1$ at the point $P(1,0,1)$ is $(x-1) + 2y + (z-1) = 0$.

(c) The electrical potential (voltage) in a certain region of space is given by the function $V(x,y,z) = 5x^2 - 3xy + xyz$.

(*i*) Find the rate of change of the voltage at point (3, 4, 5) in the direction of the vector $\underline{a} = \underline{i} - \underline{j} + \underline{k}$.

(ii) In which direction does the voltage change most rapidly at point (3, 4, 5)?

(iii). What is the maximum rate of change of the voltage at point (3, 4, 5)?

4. (a) State Gauss' Divergence Theorem.

(b) Verify the above theorem considering the vector field $\underline{F} = x^2\underline{i} + z\underline{j} + y\underline{k}$ taken over the region bounded by the planes $x = 0,\ x = 1,\ y = 0\ ,\ y = 3, z = 0$ and $z = 2$.

5. (a) (i) State Stokes' Theorem.

(ii) Verify Stokes' Theorem considering the vector field $\underline{F} = 2y\underline{i} - x\underline{j} + xz\underline{k}$ over the hemisphere S defined by $x^2 + y^2 + z^2 = 4;\ z \geq 0$ and C is its boundary.

(b) A solid whose outside is in the form of a paraboloid, given in terms of cylindrical polar coordinates by $z = 2r^2, 0 \leq z \leq 2$. Show that the volume of the solid is 16π.

6. (a) Suppose that S is a plane surface lying in the xy –plane and bounded by a closed curve C. If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$.

(b) Verify the above result for the integral $\oint_C [2xydx + (x + y)dy]$, where C is the path from (0, 0) to (1. 1) along the curve '$y = x^3$ and from (1, 1) to (0, 0) along the curve $y = x$ oriented in the counterclockwise direction.