**The Open University of Sri Lanka**
**Department of Mathematics**
**Advanced Certificate in ScienceProgramme**
**MYF2519/MHF2519 - Combined Mathematics I– Level 2**

**Final Examination 2021/22**

**Date:24-09-2022** **From 9:30am. To 12:30pm.**

Answer **All** Questions in **Part A** and Answer **Five** Questions in **Part B.**

## PART A

1. (a) Find the domain, range and codomain of the function

$$y = \frac{2x}{x^2 - 4}, \quad x \neq \pm 2.$$

(b) Sketch the graph of the above function.

2. (a) The functions $f(x)$ and $g(x)$ are defined by $f: x \to x^2$ and $g: x \to x - 1$
Find the following:

(i) $f \circ g(x)$
(ii) $g \circ f(x)$

(b) The functions $f$ and $g$ are defined as $f: x \to e^{2x}$ and $g: x \to x + 1$.

(i) Calculate $f^{-1}(3) \times g^{-1}(3)$.
(ii) Show that $(f \circ g)^{-1}(3) = ln\sqrt{3} - 1$.

3. If the roots of the equation $ax^2 + bx + c = 0$ are $\alpha$ and $\beta$, find the quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ where $\alpha, \beta \neq 0$.

4. Prove that $log_{a^n} x^m = \frac{m}{n} log_a x$. **Hence,** show that

$$log_a x + log_{a^2} x^2 + log_{a^3} x^3 + \cdots + log_{a^{2022}} x^{2022} = log_a x^{2022}$$

5. Find the equation of the straight line through the point $(-1, 3)$, perpendicular to the line $4x + 3y + 1 = 0$.

6. Solve the equation $3^{2x} + 3^x - 12 = 0$.

7. Show that $(x-1)$ is a factor of $x^3 - 2x^2 - x + 2$ . **Hence,** find the other factors of $x^3 - 2x^2 - x + 2$ .

8. If $p, q > 1$, prove that the roots of the equation

$(x-1)(2x-p-q) + (x-p)(x-q) = 0$ are real and distinct.

9. (a) If $\tan(x+y) = 33$ and $\tan x = 3$ then show that $\tan y = 0.3$.

(b) If $\tan(\theta/2) = t$ then show that $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ .

**Hence**, solve the equation $\sqrt{3}\cos\theta - \sin\theta = 1$.

10. If the roots of the quadratic equation $x^2 - px + q = 0$ are $\tan A$ and $\tan B$, then find $\sin^2(A+B)$.

## PART B

11. (a) If $x^2 + px + 1$is a factor of $ax^3 + bx + c$ then show that $a^2 - c^2 = ab$.

(b) If $x + 2$ is a factor of $(x+1)^7 + (2x+k)^3$ find the value of $k$.

(c) When the cubic expression $ax^3 + bx + c$ is divided by $x+1$, $x-1$ and $x-2$ the remainders are respectively 4, 0 and 4. Find the values of $a, b$ and $c$.

12. (a) If the roots of the equation $x^2 - 2(a-1)x + 2a + 1 = 0$ are positive find the value of $a$.

(b) The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\alpha$ and $\beta$. Find the roots of the equation $a^3x^2 + abcx + c^3 = 0$ in terms of $\alpha$ and $\beta$.

(c) The roots of the quadratic equation $x^2 - p(x+1)x - c = 0$ are $\alpha$ and $\beta$.

Show that $(\alpha+1)(\beta+1) = 1 - c$.

**Hence,** show that $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = 1$.

13. (a) Sketch the graph of $y = \tan x, 0 \leq x \leq 2\pi$. On the same graph sketch the line $y = \pi - x$.

(b) Consider the equation $x + \tan x = \pi$. Denote by $x_0$ the solution of the equation in the interval $\left(0, \frac{\pi}{2}\right)$.

(i) Find in terms of $x_0$ and $\pi$, the remaining solutions of the given equation in the interval $[0, 2\pi]$.

(ii) How many solutions does the equation $x + \tan x = \pi$, have for $x \in \mathbb{R}$?

(c) Given that $\cos A = c$ and $\sin A = s$.

(i) Write down the values of $\cos\left(\frac{\pi}{2} - A\right)$ and $\sin\left(\frac{\pi}{2} - A\right)$. **Hence**, show that

$$\tan\left(\frac{\pi}{2} - A\right) = \frac{1}{\tan A}$$

(ii) Given that $\tan A + \tan\left(\frac{\pi}{2} - A\right) = \frac{4}{\sqrt{3}}$, find possible values of $A$.

(iii) **Hence**, find the values of $A \in \left(0, \frac{\pi}{2}\right)$ that satisfy the equation given in part (ii)

14. With the usual notation for a triangle $ABC$, show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Prove that

(a) $b\sin\left(\frac{B}{2} + C\right) = (c + a)\sin\frac{B}{2}$.

(b) $\frac{\cot\frac{C}{2} + \cot\frac{A}{2}}{\cot\frac{B}{2}} = \frac{2b}{a+c-b}$.

(c) $(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0$.

15. (a) Find the general solution of the following equations:

(i) $cos3\theta + cos\theta = sin2\theta$

(ii) $\sqrt{3}sin\theta - cos\theta = \sqrt{2}$

(b) If the inverse functions take the principal values prove that

$$tan^{-1}\frac{3}{4} + tan^{-1}\frac{4}{3} = \frac{\pi}{2}.$$

(c) Find the maximum and minimum values of the expression

$$y = 11\cos^2 x + 16 \sin x \cos x - \sin^2 x.$$

16. (a) Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$. Prove that the length $PQ$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**Hence,** if $A \equiv (ap^2, 4ap)$ and $B \equiv (aq^2, 4aq)$ are given points such that $p > q$, show that $AB = a(p-q)\sqrt{(p+q)^2 + 16}$.

(b) Let $P \equiv (1, -2)$, $Q \equiv (2, 3)$, $R \equiv (-3, 2)$ and $S \equiv (-4, -3)$. Find the gradients of *PQ*, *QR*, *RS* and *SP*. Also find the lengths of *PR* and *QS*.

**Hence,** show that *PQRS* is a rhombus.

(c) The coordinates of the vertices of the triangle *ABC* are given by $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$. Show that the area of the triangle *ABC* is given by

$$\frac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\}.$$

**Hence,** find the area of the quadrilateral *ABCD* with vertices $A \equiv (0, 2)$, $B \equiv (4, 3)$, $C \equiv (1, 5)$ and $D \equiv (-1, -2)$.

17. (a) The point $C \equiv (\bar{x}, \bar{y})$ divides the line joining the points $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ internally with ratio $m:n$. Show that

$$\bar{x} = \frac{nx_1 + mx_2}{n+m} \text{ and } \bar{y} = \frac{ny_1 + my_2}{n+m}.$$

**Hence,** justify that the coordinates of the mid-point of *AB* is given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

(b) The coordinates of the centre and a vertex of a square are $(2, -1)$ and $(-1, 1)$ respectively. Find the coordinates of its other vertices.