The Open University of Sri Lanka
Department of Mathematics
Advanced Certificate in Science Programme
MYF2521/MHF2521 - Combined Mathematics 3 - Level 2
Final Examination 2021/22

Date: 17-09-2022 **From: 09:30 am. To 12:30 pm.**

Answer **All** Questions in **Part A** and Answer **Five** Questions in **Part B.**

PART A

1. Using the Principle of Mathematical Induction, prove that

$$\sum_{r=1}^{n}(2r+1) = n(n+2), \text{ for all } n \in \mathbb{N}.$$

2. Find the set of real values of x for which $\frac{1}{2}|x-1| > |x-4|$.

3. How many different numbers with five digits can be made from the digits 1, 2, 3, 4 and 5, if each digit is used only once.

How many of these numbers

(i) are even numbers?
(ii) have the digits 3 and 4 next to each other?

4. Using the binomial expansion for a positive integer index, show that

$$\left(1+\sqrt{3}\right)^6 + \left(1-\sqrt{3}\right)^6 = 416$$

Hence, find the integer part of $\left(1+\sqrt{3}\right)^6$.

5. Sketch on the same Argand diagram, the loci of points representing complex number z satisfying

(i) $|z-i| = 1$
(ii) $Arg(z-i) = \frac{\pi}{6}$.

Find the complex number represented by the point of intersection of these loci in the form $r(\cos\theta + i\sin\theta)$ where $r > 0$ and $0 < \theta < \frac{\pi}{2}$.

6. If $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ show that $A^2 - 3A = -I$. **Hence,** find A^{-1}.

7. Evaluate the following limits.

(a) $\lim_{x \to -3} \frac{x^4 - 81}{x^3 + 27}$

(b) $\lim_{x \to 0} \frac{sin3x}{sin2x}$

(c) $\lim_{x \to \infty} \frac{x^4 - 1}{3x^4 + x^2 + 1}$

8. Differentiate $y = ax^2 + bx + c$ using the first principles.

The curve defined by y passes through the point (0, 1) and the gradient of the curve at $\left(-\frac{1}{4}, \frac{7}{8}\right)$ is zero. Find the values of a, b and c.

9. A curve C is given by the parametric equation $x = 3\sin^2(\theta/2)$ and $y = \sin^3\theta$ for $0 < \theta < \frac{\pi}{4}$. Show that $\frac{dy}{dx} = \sin 2\theta$.

If the gradient of tangent at a point P on C is $\frac{\sqrt{3}}{2}$, find the value of the parameter θ corresponding to P.

10. Find the area of the region on the rectangular Cartesian plane enclosed by the curves $y = x^2$ and $y = 2x - x^2$.

PART B

11. (a) Write the general term of the series $\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots$. If $f(r) = \frac{1}{r^2}$, derive an expression for $f(r) - f(r+1)$.

Hence, find the sum of the first n terms of the series $\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots$.

Find the sum to infinity of this series and find whether the series is convergent.

(b) Show that $\frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$ and $\frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$.

Hence, find the value of $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^4 + \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^4$.

12. (a) The complex numbers z_1=2+2i and $z_2 = 1 + \sqrt{3}i$ are represented in the Argand diagram by the points A and B. O is the origin.

(i) Mark $z_1$and z_2 in the Argand diagram.

(ii) Find the lengths of OA, OB and AB.

(iii) What type of a triangle is OAB?

(iv) If $OABC$ is a rectangle, what is the complex number represented by C?

(b) If $z = \cos\theta + i\sin\theta$, for positive values of n,

show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$.

Hence, show that

$$32cos^6\theta = \cos 6\theta + 6cos4\theta + 15cos2\theta + 10 \text{ and}$$

$$32sin^6\theta = -\cos 6\theta + 6cos4\theta - 15cos2\theta + 10\ .$$

13 Differentiate following functions with respect to x.

(a) $y = (x^2 + 1)(3x^2 - 7)$

(b) $y = \frac{\sqrt{x}+\sqrt{5}}{\sqrt{x}-\sqrt{5}}$

(c) $y = \sin^2 x + \tan^2 x$

(d) $y = x^{3x}$

14. (a) Find the stationary points of the curve $y = \frac{x-2}{(x-1)(x+2)}$ and determine whether they are relative maximum points, relative minimum points or points of inflexion.

Hence, sketch the graph of the function

(b) Show that the maximum volume of a cylinder that can be contained in a cone of height $h$and radius r such that the axes of the cylinder and the cone coincide, is $\frac{4}{27}\pi r^2 h$.

15. (a) If $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin^2 x + \cos^2 x} dx$ and $J = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin^2 x + \cos^2 x} dx$ then using suitable substitution prove that $I = J$.

Hence, evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin^2 x + \cos^2 x} dx.$

(b) Evaluate following integrals.

(i) $\int \frac{2}{(x+1)(x^2+1)} dx$

(ii) $\int \frac{\cos x}{\sqrt{\sin x}} dx$

(iii) $\int x e^x dx$

16. (a) Find the volume generated, when the part of the curve $y^2 = 4ax$ in the interval

$0 \leq x \leq a$ is rotated by four right angles about the x-axis.

(b) A is a matrix such that $A^{-1} = -A$.

(i) Show that $A^2 = -I$.

(ii) Find the value of k such that $A^4 + A^3 + A^2 + A + A^{-1} = kA$.

(iii) If $A = \begin{pmatrix} a & -a \\ 2 & -a \end{pmatrix}$ and $A^{-1} = -A$, find the value of a.

17. If the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ intersect orthogonally show that $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

Find the equation of the circle through the origin which intersect the circles

$x^2 + y^2 - 6x + 8 = 0$ and $x^2 + y^2 - 2x - 10y - 4 = 0$ orthogonally.