00199

The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc. / B. Ed Degree Programme

Department	**: Mathematics**
Level	**: 04**
Name of the Examination	**: Final Examination**
Course Title and - Code	**: Vector Calculus – ADU4302/ADE4302**
Academic Year	**: 2023/24**
Date	**: 06.10.2023**
Time	**: 9.30 a.m. To 11.30 a.m.**
Duration	**: Two Hours.**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **(6)** questions in **(2)** pages.
3. Answer any **(4)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Involvement in any activity that is considered as an exam offense will lead to punishment
6. Use blue or black ink to answer the questions.
7. Clearly state your index number in your answer script.

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1. (a) State and sketch the domain of the function $f(x,y) = \ln(x^2 + y^2 - 1)$.

(b) Sketch at least three level curves of the function $f(x,y) = \ln(x^2 + y^2 - 1)$.

(c) Find the following limits if they exist

(i) $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$, (ii) $\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+y^2}$.

justifying your answer.

(d) Discuss the continuity of the following function at (0, 0).

$$f(x,y) = \begin{cases} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}, & \text{if} \quad (x,y) \neq (0,0) \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

(You may use your conclusion regarding c(i).)

2. (a) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D. Explain briefly how you could determine its nature.

(b) Find the maximum and minimum values of the function $f(x,y) = 5x^3 - 3xy + 5y^3$ and determine their nature.

(c) If the velocity of a fluid at the point (x,y,z) is given by $\underline{v} = (ax + by)\underline{i} + (cx + dy)\underline{j}$ find the conditions on the constants a, b, c and d in order that $div\ \underline{v} = 0$ and $Curl\ \underline{v} = \underline{0}$.
Verify that in this case $\underline{v} = \frac{1}{2} grad(ax^2 + 2bxy - ay^2)$.

3. (a) Prove that grad ϕ is a vector normal to the contour surface $\phi(x,y,z) = c$, where c is a constant.

(b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is given by $(x - x_0)\left(\frac{\partial F}{\partial x}\right)_P + (y - y_0)\left(\frac{\partial F}{\partial y}\right)_P + (z - z_0)\left(\frac{\partial F}{\partial z}\right)_P = 0$.

(ii) Using the above result, show that the equation of the tangent plane to the surface $2z = x^2 + y^2$ at the point $P(1,3,5)$ is $x + 3y - z = 5$.

(c) Show that if $\underline{r} = x\underline{i} + y\underline{j} + z\,\underline{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$ then

(i) $\nabla(\ln r) = \frac{\underline{r}}{r^2}$,

(ii) $\nabla\left(\frac{1}{r}\right) = -\frac{\underline{r}}{r^3}$,

(iii) $(\underline{a} \cdot \underline{\nabla})\underline{r} = \underline{a}$, where $\underline{a}$ is a constant vector.

4. (a) State Gauss' Divergence Theorem.

(b) Verify the above theorem considering the vector field $\underline{F} = 2xz\underline{i} + yz\underline{j} + z^2\underline{k}$ where S is the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$ and $z = 0$ plane.

5. (a) State Stokes' Theorem.

(b) Verify the above theorem considering the vector field $\underline{F} = (y - z + 2)\underline{i} + (yz + 4)\underline{j} - xz\underline{k}$ where S is the surface of the cube $x = 0,\ y = 0,\ z = 0\ , x = 2, y = 2$ and $z = 2$ above the xy-plane.

6. (a) Suppose that S is a plane surface lying in the xy –plane and bounded by a closed curve C. If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$.

(b) Verify the above result for the integral $\oint_C \{(x^2 + y^2)dx + (x + 2y)dy\}$, where C is the path along the line $y = 0$, along the curve $x^2 + y^2 = 4$ and along the line $x = 0$ oriented in the counterclockwise direction.

[illegible]