00155

The Open University of Sri Lanka

B.Sc/B.Ed. Degree Programme – Level 04

Final Examination – 2023/2024

Pure Mathematics

PEU4300 – Real Analysis 1

Duration: - Two Hours.

Date: - 07.10.2023 **Time: - 01.30 p.m.-03.30 p.m.**

Answer Four Questions only.

(01) (a) Using the definition of limit, Prove each of the following.

(i) $\lim_{n\to\infty} \frac{4n^2-8}{2n^2-n} = 2$, (ii) $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1$.

(b) Show that

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$$

(c) Let $\langle x_n \rangle$ be a convergent sequence such that $\lim_{n\to\infty} x_n = 1$. Prove that

$\lim_{n\to\infty} x_n^k = 1$ for each $k \in \mathbb{N}$.

Does it follow that $\lim_{n\to\infty} x_n^n = 1$? Justify your answer.

(02) (a) Let $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7 + u_n}$ for each $n \in \mathbb{N}$. Prove that

(i) $\langle u_n \rangle$ is monotonically increasing,

(ii) $\langle u_n \rangle$ is bounded above,

(iii) $\langle u_n \rangle$ is convergent and $\lim_{n\to\infty} u_n = \frac{1+\sqrt{29}}{2}$.

(b) Let $\langle x_n \rangle$ be a monotonically increasing sequence. Prove that

$\left\langle \frac{1}{n} \sum_{k=1}^{n} x_k \right\rangle$ is also a monotonically increasing sequence.

(03) (a) Using the definition of a sequence diverges to infinity, prove that the sequence

$\left\langle \frac{2n^3+3}{n^2+1} \right\rangle$ diverges to ∞.

(b) Let $\langle x_n \rangle$ be the sequence given by $x_n = \begin{cases} \frac{1}{n^2} ; n \text{ is even} \\ \frac{1}{n} ; n \text{ is odd} \end{cases}$

(i) Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

(ii) Show that the subsequences $\langle x_{2n} \rangle$ and $\langle x_{2n-1} \rangle$ converge.

(iii) Using Sandwich theorem show that $\langle x_n \rangle$ converges.

(c) Find the limit superior and limit inferior of each of the following sequences.

(i) $\langle x_n \rangle = (-2)^{-n}\left(1 + \frac{1}{n}\right)$ (ii) $\langle y_n \rangle = (-1)^n\left(1 + \frac{1}{n}\right)^2$

(04) (a) Write down the definition of Cauchy sequence.

Prove directly from the definition that the sequence given by $a_n = \frac{n+3}{2n+1}$ for Each $n \in \mathbb{N}$ is a Cauchy sequence.

(b) Show that $\langle \sum_{k=1}^{n} \frac{1}{k} \rangle$ is not a Cauchy sequence.

(c) Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

(d) Suppose $\sum_{n=1}^{\infty} ar^{n-1}$ is a convergent geometric series. Prove that

$\sum_{n=m}^{\infty} ar^{n-1}$ is a convergent geometric series and $\sum_{n=m}^{\infty} ar^{n-1} = \frac{ar^{m-1}}{1-r}$ for

each $m \in \mathbb{N}$.

(05) Determine the convergence or divergence of each of the following series:

(i) $\sum_{n=1}^{\infty}\left(\frac{1}{n^3} + \frac{1}{2^n}\right)$, (ii) $\sum_{n=1}^{\infty}\left(\sqrt{1+n^4} - n^2\right)$,

(iii) $\sum_{n=1}^{\infty}\left(\frac{1}{3\sqrt{n}+5} - \frac{1}{5\sqrt{n}+7}\right)$, (iv) $\sum_{n=1}^{\infty} \frac{n^{\frac{5}{6}} + n^{\frac{5}{8}} + 1}{n^2+3\sqrt{n}-1}$,

(v) $\sum_{n=1}^{\infty} \frac{5^n}{2^n+5}$.

(06) (a) Determine the radius of convergence of each of the following power series:

(i) $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n$,

(ii) $\sum_{n=1}^{\infty} \left(\frac{n+2}{2n}\right)^n x^n$.

(b) Find whether each of the following series is conditionally convergent, absolutely Convergent or divergent:

(i) $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\sqrt{n^2+1} - n\right)$,

(ii) $\sum_{n=1}^{\infty} \frac{7-3n}{n^3+1}$,

(iii) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+1}{n+2}\right)$.