00054

The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2023/2024
Pure Mathematics - Level 05
PEU5305 – Complex Analysis I
Duration: - Two hours

Date: - 18-10-2023 **Time: - 09.30 a.m. – 11.30 a.m.**

Answer **FOUR** Questions **ONLY**.

1.

a) Determine where the function $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$ is differentiable and where it is analytic.

b) Prove that the function $u(x, y) = \sinh x \sin y$ is harmonic.
Find a function $v(x, y)$ such that $f(z) = u + iv$ is analytic in $\mathbb{C}$, and express $f(z)$ in terms of z.

c) Let $f(z) = u + iv$ be analytic in a region G. Show that if $u + 2v = 5$ in G, then $f(z)$ is constant in G.

2.

a) Solve each of the following equations:
   i. $e^{2z} = 1 + i$,
   ii. $\sin z = 2$,
   iii. $\cosh z = 1$.

b) Prove that $\cos z = 0$ if and only if $z = (2n+1)\frac{\pi}{2}$, where n is an integer.

c) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Show that $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

3.

a) Let C be the unit circle $z = e^{i\theta}, -\pi \leq \theta \leq \pi$. Show that, for any real constant a,

$\int_C \frac{e^{az}}{z} dz = 2\pi i.$

Deduce that $\int_0^{\pi} e^{a\cos\theta} \cos(a\sin\theta) d\theta = \pi.$

b) State Cauchy's Integral Formula for Higher Derivatives.

c) Using Cauchy's Integral Formula for Higher Derivatives, evaluates the integral

$\int_C \frac{e^z}{(4z-\pi i)^4} dz,$ where C is the square with vertices at $\pm 1 \pm i$, oriented counterclockwise.

4.

a) State Green's Theorem in the plane.

b) Evaluate the line integral $\int_C (2x - 3y)dx + 5x\, dy$, where C is the unit circle given by

$x = \cos\theta, y = \sin\theta;\ 0 \leq \theta \leq 2\pi.$

i. Using Green's Theorem in the plane.

ii. Using the parameterization of C.

c) Show that the area A bounded by a simple closed contour C is given by

$A = \frac{1}{2}\int_C x\, dy - y\, dx.$

Using this formula, find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

5.

a) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$.

b) Let $f(z) = \frac{1}{(3z+1)(z-1)}$. Find the Laurent series expansion of $f(z)$ in each of the following annuli:

i. $\frac{1}{3} < |z| < 1,$

ii. $0 < \left|z + \frac{1}{3}\right| < \frac{4}{3}.$

c) Show that the function $f(z) = \frac{\cos z}{z^2}$ has a double pole at $z = 0$.

d) Find and classify the singularities of the function $f(z) = \frac{e^z}{(z+3)(z-i)^3}$.

6.

a) State Cauchy's Residue Theorem.

b) Using Cauchy's Residue Theorem, calculate each of the following contour integrals:

i. $\int_C \frac{1}{(z-1)^2(z+1)(z-2)} dz$, where C is the circle $|z-1| = 3$, oriented counterclockwise.

ii. $\int_C \tan z \, dz$, where C is the circle $|z| = 3$, oriented counterclockwise.

c) Using Cauchy's Residue Theorem, show that $\int_0^{\pi} \frac{1}{(5+3\cos\theta)^2} d\theta = \frac{5\pi}{64}$.