The Open University of Sri Lanka

B.Sc./B.Ed., Continuing Education Degree Programme

Final Examination-2023/2024

ADU5302/ADE5302-Mathematical Methods

Applied Mathematics -Level 05

DURATION: Two (02) Hours

Date: 16.10.2023 **Time:1.30p.m. - 3.30p.m.**

Answer FOUR questions only.

1.(a) Find the Laplace transform $L(t)$ of

(i) $\cosh at \sin at$. (ii) $\sin 2t \cos 3t$.

(b) Find the inverse Laplace transform of $\dfrac{s+4}{s(s-1)(s^2+4)}$.

(c) Using the convolution theorem, find the inverse Laplace transform of

$$\frac{s^2}{(s^2+a^2)^2}.$$

(d) Solve the following boundary value problem using the Laplace transform method

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=0 \quad \text{subject to} \quad y(0)=-1,\ y'(0)=7.$$

2. Consider the boundary value problem,

$$\frac{d^2y}{dx^2}+\mu y=0$$

$$y(0)=y'(0),\ \ y(\pi)=y'(\pi).$$

(a) Show that this is a Sturm-Liouville problem.

(b) Show that when $\mu > 0$, the eigenfunctions of the problem is given by

$y_n = B_n(n\cos nx + \sin nx)$ for $n = 1,2,3,.....$ Here B_n is an arbitrary constant.

(c) Verify that the eigenfunctions are mutually orthogonal in the interval $0 \le x \le \pi$.

(d) Show that if $B_n = \sqrt{\dfrac{2}{\pi(n^2+1)}}$, the eigenfunctions are orthonormal in the interval

$0 \le x \le \pi$.

$$\left(\begin{array}{l}\text{You may use the following results without proof :}\\ \int_0^{\pi} \sin mx \sin nx\,dx = 0;\ \int_0^{\pi} \cos mx \cos nx\,dx = 0;\ \int_0^{\pi} \sin mx \cos nx\,dx = 0;\ \text{when } m \ne n.\end{array}\right)$$

3. (a) Determine the Fourier series for the function given below:

$$f(x) = \begin{cases} x^2 & 0 < x \le 1 \\ 2 - x & 1 < x < 2 \end{cases}$$

and $0 \le x \le 2$.

(b) Let $f(x)$ be a function defined in the interval $0 < x < \pi$ by $f(x) = x^2$.

Find the Fourier sine series and the Fourier cosine series of $f(x)$.

4. (a) The Gamma function denoted by $\Gamma(p)$ corresponding to the parameter p is defined by the

improper integral $\Gamma(p) = \int_0^{\infty} e^{-t} t^{p-1} dt, \quad (p > 0).$

(i) Evaluate $\dfrac{\Gamma(3).\Gamma(2.5)}{\Gamma(4.5)}$ and $\Gamma(-3.5)$.

(ii) Evaluate $\int_0^{\infty} 3^{-4x^2} dx.$ using Gamma function.

(b) The Beta function denoted by $\beta(p,q)$ is defined by $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx,$

where p and q are positive parameters.

(i) Use Gamma function and Beta function to evaluate the following integral:

$$I=\int_0^2 x\sqrt{8-x^3}\,dx.$$

(ii) Show that $\left[\int_0^{\frac{\pi}{2}}\sqrt{\sin\theta}\,d\theta\right]\left[\int_0^{\frac{\pi}{2}}(\sin\theta)^{-\frac{1}{2}}\,d\theta\right]=\pi.$

5. Let $J_p(x)$ be the Bessel function of order p given by the expansion

$$J_p(x)=x^p\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+p}.m!\Gamma(p+m+1)}.$$

Prove each of the following results:

(a) $J_3(x)+3J_0'(x)+4J_0'''(x)=0$; where $'''$ denotes a standard notation.

(b) $\int x[J_0^2(x)]dx=\frac{x^2}{2}[J_0^2(x)+J_1^2(x)]$.

(c) $J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x.$ and $J_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right).$

$\left(\text{Hint 1}: J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x.\right)$

[Hint 2: You may use the following recurrence relations, if necessary, without proof.

$$\frac{d}{dx}\{x^pJ_p(x)\}=x^pJ_{p-1}(x)$$

$$\frac{d}{dx}\{x^{-p}J_p(x)\}=-x^{-p}J_{p+1}(x).$$

$$J_p'(x)=\frac{p}{x}J_p(x)-J_{p+1}(x)$$

$$J_p'(x)=\frac{p}{x}J_p(x)-J_{p+1}(x)]$$

6. The Rodrigue's formula for the n^{th} degree Legendre polynomial denoted by $P_n(x)$ is given as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

$P_n(x)$ is also given by the sum

$$P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!}.x^{n-2m}, \quad n = 0,\ 1,\ 2,....$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$, whichever is an integer.

(a) Prove that $(n+1)p_{n+1} = (2n+1)xp_n - np_{n-1}$.

(Hint: You may use the following recurrence relations, if necessary, without proof.)

$$P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x),$$
$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x),$$
$$xP_n'(x) = nP_n(x) + P_{n-1}'(x),$$
$$(1-x^2)p_{n-1}' = n(xp_{n-1} - p_n),$$
$$(x^2-1)(p_n') = n(xp_n - p_{n-1}).$$

(b) Using Rodrigues formula find $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$.

(c) Using part (b) above write down the function $2x^3 + 5x^2 + x + 1$ in terms of Legendre polynomials.

(d) Prove that $\int_{-1}^{1}(1-x^2)p_m'\,p_n'dx = \begin{cases} 0 & ; \quad \text{if } m \neq n \\ \dfrac{2n(n+1)}{2n+1} & ; \quad \text{if } m = n \end{cases}$

(Hint: Multiply Legendre's equation for p_m by p_n, integrate -1 to 1 and use orthogonality.)

The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme - Level 05
Final Examination-2023/2024
Applied Mathematics
ADU5307/ADE5307 –Numerical Methods

Duration: Two Hours

Date: 08. 10. 2023 **Time: 1.30 p.m. –3.30 p.m.**

Answer **any (4)** questions only.

1. (a) Show that the recurrence relation to find the root of a given function $y = f(x)$ using the method of False Position is given by $x_{n+1} = x_{n-1} - \frac{f(x_{n-1})}{f(x_n)-f(x_{n-1})}(x_n - x_{n-1})$ where $n = 1, 2, 3, ...$

   (b) Show that the equation $x^2 - 12 = 0$ has a root in the interval $[3, 4]$. Hence find the root correct to four decimal places using the method of False Position.

2. (a) Prove that

   (i) $E = \Delta + I,$
   (ii) $\nabla = I - E^{-1}$
   (iii) $\mu = \left(\frac{1}{2}\right)\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

   where Δ, ∇, E, I and μ are the forward difference, backward difference, the shift, identity, and average operators respectively.

   (b) Derive Lagrange's interpolation formula.

   The following table shows the population of a country during three census periods.

   Using Lagrange's interpolation formula, find the population during 1966.

Year	1951	1961	1971
Population (Million)	2.8	3.2	4.5

3. (a) State the Trapezoidal rule, Simpon's One-Third rule and Simpon's Three-Eighth rule.

Applying above three methods separately for the data given in the following table, evaluate $\int_0^{\frac{\pi}{2}} \sin x \; dx$, with $h = \pi/12$.

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$f(x) = sin\, x$	0	0.258819	0.5	0.707106	0.866025	0.965925	1

(b) Find the absolute error of each method. Hence find the method which provides higher accuracy.

4. (a) Applying the Picard's method, find the first three successive approximations to solve

$\frac{dy}{dx} = x + y$ with the initial condition $y(0) = 1$. Hence find values of $y(0.1)$ and $y(0.2)$.

(b) Using third order Runge-Kutta algorithm, solve $\frac{dy}{dx} = -y$ at $x = 0.1$, subject to the initial condition $y(0) = 1$.

5. (a) Applying Taylor series method of third order for the differential equation

$\frac{dy}{dx} = x^2y - 1$ subject to the initial condition $y(0) = 1$, evaluate $y\ (0.1)$ and $y\ (0.2)$ correct to four decimal places.

(b) Applying Taylor series method of the third order for the following differential equations

$$\frac{dx}{dt} = x + y + t,$$

$$\frac{dy}{dt} = 2x - t$$

subject to the initial conditions $x = 0$, $y = 1$, and $t = 1$, evaluate $y\ (0.1)$ and $x(0.1)$ correct to four decimal places.

6. (a) Applying Euler's method, solve $\frac{dy}{dx} = \frac{y-x}{y+x}$ subject to the initial condition $y(0) = 1$ with five steps, find the value of y at $x = 0.1$

(b) Applying the modified Euler's method, solve the equation $y' = x + y$ subject to the initial condition $y(0) = 0$ for $x = 0.6$ by taking $h = 0.2$.

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The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination - 2023/2024
Pure Mathematics - Level 05
PEU5300 - Riemann Integration

Duration: Two Hours

Date: 11.10.2023 **Time: 9.30 a.m. - 11.30 a.m.**

ANSWER FOUR QUESTIONS ONLY.

1. (a) Let f be a bounded function defined on an interval $[a, b]$. Define the upper sum $U(P, f)$ and the lower sum $L(P, f)$ of f with respect to a partition P of $[a, b]$.

   (b) Let f and g be bounded functions on $[a, b]$ and let $P \in P[a, b]$. Then, prove that
   $$L(P, f) + L(P, g) \leq L(P, f + g) \leq U(P, f + g) \leq U(P, f) + U(P, g).$$

   (c) Let $P = \{0 = x_0, x_1, x_2, \ldots, x_{i-1}, x_i, \ldots, x_n = 1\}$ be an equally spaced partition of $[0,1]$ with the norm $\Delta x = \frac{1}{n}$. Compute $U(P, f)$ and $L(P, f)$ for the function $f(x) = x^2\ ; 0 \leq x \leq 1$. Deduce that f is Riemann integrable on $[0,1]$.
   (You may use $\sum_1^n i^2 = \frac{n}{6}(n + 1)(2n + 1)$)

2. (a) Let f and g be bounded Riemann integrable functions on $[a, b]$ such that for each $x \in [a, b]$, $f(x) \leq g(x)$. Prove that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

   (b) (i). Let f be a Riemann integrable function on $[a, b]$ such that for each $x \in [a, b]$, $m \leq f(x) \leq M$ for some $m, M \in \mathbb{R}$. Prove that $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$.

   (ii). Let $f(x) = \left(x + \frac{1}{x}\right)^x, x \in [1,2]$. Assuming that f is Riemann integrable on $[1, 2]$, prove that $2 \leq \displaystyle\int_1^2 \left(x + \frac{1}{x}\right)^x dx \leq \frac{25}{4}$.

3. State the Riemann's Criterion for integrability of a bounded function f defined on $[a, b]$. Using the Riemann's Criterion prove that,
   (i). If f is a continuous function on a closed and bounded interval $[a, b]$, then f is Riemann Integrable on $[a, b]$.
   (ii). $f(x) = \begin{cases} 2, & x \in [2,3] \\ 3, & x \in (3,4] \end{cases}$ is Riemann integrable on $[2,4]$.
   (iii). $g(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$ is not Riemann integrable on $[0,1]$.

4. (a) State the Fundamental Theorem of Calculus (FTC).

(b) Suppose u is continuously differentiable on $[a, b]$ and $u'(t) \neq 0$ for each $t \in [a, b]$.
Let $v: u([a, b]) \to \mathbb{R}$ be the function inverse to u. Suppose f is continuous on $u([a, b])$, then prove that

$$\int_a^b f[u(t)]dt = \int_{u(a)}^{u(b)} f(x)v'(x)dx.$$

Hence, show that

(i). $\int_1^9 \frac{1}{(\sqrt{t}+1)} dt = 2(2 - ln\ 2)$

(ii). $\int_1^8 \frac{1}{(1+\sqrt[3]{t})} dt = \frac{3}{2} + 3\ln\left(\frac{3}{2}\right)$

5. (a) State and prove the Mean-Value Theorem for integrals.
Let $f(x) = x^4 + \cos x,\ \ x \in [1,3]$. Prove that there exists $c \in [1,3]$ such that $c^4 + \cos c = \frac{242}{10} + \frac{1}{2}(\sin 3 - \sin 1)$.

(b) Using the properties of Riemann sums, find two different integrals over the interval $[0, 3]$ and $[-1, 2]$ that are represented by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[4 - \frac{9i^2}{n^2}\right]\left(\frac{3}{n}\right).$$

Hence find the above limit.

6. Explaining why the following integrals are improper, determine the convergence of each of them and evaluate them if they converge.

(a) $\int_{-1}^{1} \frac{1}{\sqrt{|x|}} dx$

(b) $\int_0^{\infty} \frac{1}{\sqrt{x}(x+1)} dx$

(c) $\int_1^{\infty} \frac{1}{(x-2)^2(x-4)^2} dx$

(d) $\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx$