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The Open University of Sri Lanka
Department of Mathematics
Advanced Certificate in Science Programme
MYF2521/MHF2521 - Combined Mathematics 3 - Level 2
Final Examination 2023/24

Date: 02-09-2023 **From: 09:30 am. To 12:30 pm.**

Answer **All** Questions in **Part A** and Answer **Five** Questions in **Part B.**

PART A

1. Using the **Principle of Mathematical Induction**, prove that $23^n - 1$ is divisible by 11 for all positive integers n.

2. Show that $\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$. **Hence** find $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$.

3. A committee of 5 students to be chosen from 6 boys and 9 girls. Find the number of ways that this can be done if

   (a) there are no restrictions on the choice of students for the committee.

   (b) the committee must include 3 boys and 2 girls.

   (c) the committee should have more girls than boys.

4. Using the binomial expansion for a positive integer index, show that $\left(3+\sqrt{5}\right)^5 + \left(3-\sqrt{5}\right)^5$ is an even number.

5. (a) Describe the locus represented by the complex equation $Arg(z-4) - Arg(z) = \frac{\pi}{3}$.
   (b) Write the cartesian equation equivalent to the complex equation $|z-2+i| = 3$.

6. If $A = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$, find the matrices X, Y and Z such that

   (a) $X = B^2 + 2A$.
   (b) $AY = B$.
   (c) $ZA = B$.

7. Evaluate the following limits.

(a) $\lim_{x\to-2} \frac{x^4-16}{x^3+8}$

(b) $\lim_{x\to0} \frac{\sin7x}{\sin2x}$

(c) $\lim_{x\to\infty} \frac{2x^3+3x^2-1}{3x^3+x+1}$

8. A curve C is given by the parametric equation $x = \sec^3\theta$ and $y = \tan^3\theta$.

Show that $\frac{dy}{dx} = \sin\theta$. Find the equation of the tangent at the point corresponding to the parameter $\theta = \frac{\pi}{4}$.

9. Find the area of the region on the rectangular Cartesian plane enclosed by the curves $y = (x-1)^2$ and $y = 2-(x-1)^2$.

10. The circle C has equation $x^2+y^2-10x+4y+11=0$. Find the coordinates of the centre and radius of C. Given that the line $y = 3x+k$, where k is a constant, is a tangent to the circle. Show that $k = -17 \pm 6\sqrt{5}$.

PART B

11. (a) Verify that $\frac{2r-1}{r(r-1)} - \frac{2r+1}{r(r+1)} = \frac{2}{(r-1)(r+1)}$.

Hence, using method of differences, prove that $\sum_{r=1}^{n} \frac{2}{(r-1)(r+1)} = \frac{3}{2} - \frac{2n+1}{n(n+1)}$.

Find the sum to infinity of this series and find whether the series is convergent.

(b) If $x+\frac{1}{x}=1$ find the values of (i) $x^5+\frac{1}{x^5}$ and (ii) $x^7+\frac{1}{x^7}$.

12. (a) The complex numbers $z_1 = \frac{5-10i}{2-i}$ and $z_2 = -5i$ are represented in the Argand diagram by the points A and B. The point O denote the origin.

(i) Mark z_1 and z_2 in the Argand diagram.

(ii) Find the lengths of OA, OB and AB.

(iii) Prove that the triangle OAB is isosceles.

(iv) Find the angle AOB.

(b) If $z = \cos\theta + i\sin\theta$, for positive values of n,

show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$.

Hence, show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \quad \text{and}$$

$$\sin 5\theta = 16\sin^5\theta - 20\,\text{sins}^3\,\theta + 5\sin\theta\,.$$

13. (a) Differentiate following functions with respect to x.

(i) $y = (2x^2 + 1)(3x^3 + 7x^2 + 2)$

(ii) $y = \frac{\cos x}{\sqrt{\cos x + \sin xx}}$

(iii) $y = \ln[\sec 3x + \tan 3x]$

(iv) $y = (\cos x)^{\sin x}$

(b) If $x = e^t \sin t$ and $y = e^t \cos t$ then show that $\frac{dy}{dx} = \frac{y-x}{y+x}$.

14. (a) Find the stationary points of the curve $y = \frac{x-1}{(x-2)(x+3)}$ and determine whether they are relative maximum points, relative minimum points or points of inflexion.

Hence, sketch the graph of the function

(b) A right circular cone of semi-vertical angle θ is circumscribed about a sphere of radius R. Show that the volume of the cone is

$V=\frac{1}{3}\pi R^3(1+\operatorname{cosec}\theta)^3\tan^2\theta$ find the value of θ when V is minimum.

15. (a) Prove that $I=\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin x}dx=\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos x}dx$. Hence, evaluate $\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos x}dx$.

(b) Evaluate following integrals.

(i) $\int\frac{2x+4}{(x-2)(x^2+4)}dx$

(ii) $\int\frac{(\tan^{-1}x)^3}{1+x^2}dx$

(iii) $\int e^x\cos x\,dx$

16. (a) Find the volume generated, when the part of the curve $y=x^2$ in the interval $1\le x\le 2$ is rotated by four right angles about the x-axis.

(b) Matrix A is given by $A=\begin{pmatrix}3 & 2\\ -2 & -1\end{pmatrix}$.

(i) Find A^2.

(ii) Find a and b such that $A^2=aA+bI$.

(iii) Find A^{-1}.

17. If the circles $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$ intersect orthogonally show that $2g_1g_2+2f_1f_2=c_1+c_2$.

Let $S_1=x^2+y^2-2x-6y+2=0$ and $S_2=x^2+y^2-5x-8y+3=0$. Find the equation of the circle through the origin which passes through the points of intersection of the two circles S_1 and S_2. Show that this circle intersects the given circle S_1 orthogonally.