THE OPEN UNIVERSITY OF SRI LANKA
Faculty of Engineering Technology
Department of Mathematics & Philosophy of Engineering

Bachelor of Technology Honors in Engineering /
Bachelor of Software Engineering Honors 037

Final Examination (2021/2022)
MHZ5355 / MHZ5375: Discrete Mathematics

Date: 23rd February 2023 (Thursday) **Time: 09:30 – 12:30**

Instruction:

- **Answer only five questions.**
- **Please answer a total of five (05) questions choosing at least one from each single section.**
- **All symbols are in standard notation and state any assumption that you made.**

SECTION – A

Q1.

i. Prove the following statements using Mathematical induction.

a) $6^n - 5n + 4$ is divisible by 5 for all $n \geq 1$; **[20%]**

b) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$ for all $n \geq 1$. **[20%]**

ii. Let a, b, and c be any integer numbers. Prove that,

a) if $c|a$ and $c|b$, then $c|(2c - 3a + 7b)$, **[10%]**

b) if $a|b$, and $b|a$ then $a = \pm b$, **[10%]**

iii.

a) Let a, b are integers and $gcd(a, b) = 1$. Show that $gcd(a + b, a - b) = 1$ or 2. **[15%]**

b) Find the $gcd(12378, 3054)$, and find the integers x, y such that $\gcd(12378,\ 3054) = 12378x + 3054y$ by using the Euclidean Algorithm. **[15%]**

c) Determine all integer solutions of the following Diophantine equation:

$$12378x_0 + 3054y_0 = 90.$$ **[10%]**

Q2.

i. Let a, b, c and d be integers. Let n and m be a positive integer.

a) If $a \equiv b(mod\, n)$ and $b \equiv c(mod\, n)$, then show that $a \equiv c(mod\, n)$; **[15%]**

b) If $a \equiv b(mod\, n)$ and $m|n$, then show that $a \equiv b\, (mod\, m)$; **[15%]**

c) Prove that $a \equiv b\, (mod\, m)$, $b \equiv a\, (mod\, m)$ and $a - b \equiv 0\, (mod\, m)$ are equivalent statements. **[15%]**

d) If $a \equiv b(mod\, n)$ and $a \equiv c(mod\, m)$, then prove that $b \equiv c(mod\, k)$ where the integer $k = \gcd(n,\ m)$. **[15%]**

ii. Solve the following system of congruence: **[40%]**

$$2x \equiv 1\, (mod\, 5)$$
$$3x \equiv 9\, (mod\, 6)$$
$$4x \equiv 1\, (mod\, 7)$$
$$5x \equiv 9\, (mod\, 11).$$

SECTION – B

Q3.

i. Determine whether the operations " $*$ " : $\mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ defined below are binary operation or not. Prove your answer if " $*$ " is a binary operation, explain the reason if " $*$ " not a binary operation. **[30%]**

a) $x * y = \sqrt{xy}$; $\quad x, y \in \mathbb{Z}$,

b) $x * y = x^2 - y^2$; $\quad x, y \in \mathbb{Z}$,

c) $x * y = \frac{x+y}{x-y}$; $\quad x, y \in \mathbb{Z}$ and $x \neq y$.

ii. Let A be set of real numbers with the operation " $*$ " defined by $x * y = x + 2xy$.

a) Is this operation associative?

b) Is this operation commutative?

Justify your answer. **[25%]**

iii. Define an abelian group $(G,\ *)$ in usual notation.
Let G be the subset of $\mathbb{Z}$ with respect to the binary operation " $*$ " defined by $a * b = a + b + 3$ for all $a, b \in G$. **[45%]**

a) Show that $(G,*)$ is a group.

b) Is the group $(G,*)$ Abelian? Justify your answer.

Q4.

i. Define a semi-group $(G, \#)$ in usual notation.
Let operations "#" : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, defined as follows.

a) "$a \# b = a^b$" for all $a, b \in \mathbb{R}$;
b) "$c \# d = 3(c+d)$" for all $c, d \in \mathbb{R}$;
c) "$e \# f = \sqrt{ef}$" for all $e, f \in \mathbb{R}^+$.

Verify that where $(\mathbb{R}, \#)$ a semi-group or not for each of the above cases. [45%]

ii. Let R={0, 1, 2, 3, 4, 5, 6} be a group under the operation $\oplus_7$. The operation $\oplus_7$ is defined by $a \oplus_7 b = r$ and $0 \leq r \leq 6$, where r is the non-negative remainder when ordinary addition $a+b$ is divided by 7. [25%]

a) Determine the identity element of R.
b) Determine the inverse of each element $a \in R$.

iii. Define a homomorphism for group in usual notation. [30%]
Let $A = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}\}$ be a group under **matrix multiplication.**
Let $H: A \rightarrow \mathbb{Z}$ be defined by $H\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.
Prove that H is a homomorphism of A on to $\mathbb{Z}$.

SECTION – C

Q5.

i. Determine which of the following are simple, by drawing each of them.

a) $H_1 = \{V_1, E_1\}$ where $V_1 = \{1, 2, 3, 4, 5, 6, 7\}$ and
$E_1 = \{\{x, y\}, 3x + 2y \text{ is odd and } x > y\}$ [10%]

b) $H_2 = \{V_2, E_2\}$ where $V_2 = \{2, 3, 4, 5, 11, 12, 13, 14\}$ and "s" and "t" are ajacent if and only if $\gcd(s, t) = 1$ and $s > t$. [10%]

c) $H_3 = \{V_3, E_3\}$ where $V_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and
$E_3 = \{\{m, n\}$: $m \times n$ is a divisible of 5 and $m \leq n\}$. [10%]

ii. Let G be a graph of 12 vertices and 17 edges, and x, y, z denoted by the number of vertices in G of degree 2 ,3 and 4 respectively. Assume that $y \geq 3$. Find all possible answers for (x, y, z). [25%]

iii. G is the graph whose adjacency matrix M is given by

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

a) Without drawing a graph of G, explain whether G is connected or not. **[15%]**

b) If $V(G) = \{a, b, c, d\}$ then find the number of paths of length four joining vertices a and c. **[10%]**

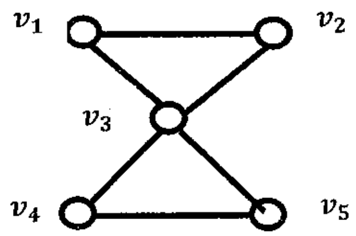
c) Draw the graph of an adjacency matrix M. **[05%]**

iv. What is the largest possible number of vertices in a graph with 2022 edges if all vertices have degree at least four? **[15%]**

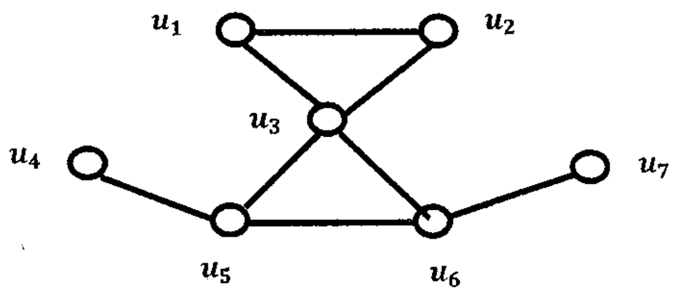
Q6.

i. Determine whether following graph are Eulerian or not. Which of them are Hamiltonian ? Justify your answers. **[20%]**

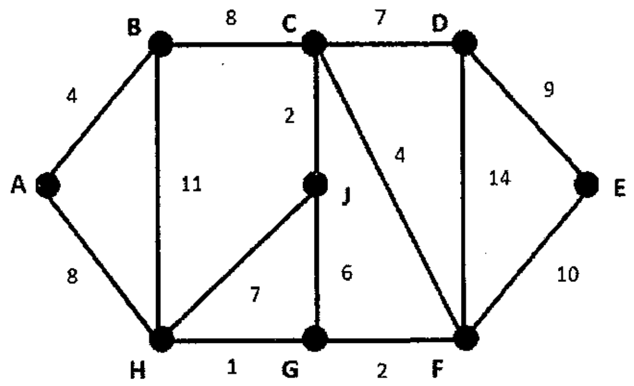
a)



b)



ii. Use Dijkstra's algorithm to find the shortest route between node A and every other node in the following network. **[60%]**



iii. Draw a tree following case: **[20%]**

"Twelve vertices exactly four of which has degree two".

SECTION – D

Q7.

i. Iterate the Eco-system growth given by the relation $t_{n+1} - \lambda t_n = 0$ starting with $t_0 = 0.3$, taking $\lambda = 0.5$ and $\lambda = 1.5$, and draw the diagrams. Hence deduce t_n as $n \to \infty$.
(At least 5 iteration steps are necessary) **[20%]**

ii. Iterate the Eco-system growth model relationship $y_{n+1} = 1.8\, y_n(1 - y_n)$, were $y_0 = 0.3$, obtain the convergent value of the model.
(Up to 4 decimal places probably) **[20%]**

iii. Consider the iterations given by $Z_{n+1} = Z_n^2$ and suppose that $Z_0 = (1.2 + i\,0.5)$. Find Z_5 and draw the graph of the iteration. **[10%]**

iv. Suppose a system with two unknowns x and y, is modeled in the form of a system of differential equations

$$\frac{dx}{dt} = 5x + 2y$$
$$\frac{dy}{dt} = 3x + 6y$$

with initial conditions $x = 3$ and $y = 2$ when $t = 0$.
Find the space values$(x_t,\ y_t)$. **[50%]**

Q8.

i. Let $L = \{1, 12, 21\}$ and $M = \{21, 212, 121\}$ be languages. Find the concatenations LM and ML. **[20%]**

ii. Suppose $L(G) = \{a^m b^n \mid m > 0 \; and \; n \geq 0\}$. Find out the grammar G which produces L. **[20%]**

iii. Show that the string $\{a^n b^m c^m d^n \mid n > 0, m > 0, n \neq m\}$ is a sentence generated by the grammar and starting symbol S and production P. **[20%]**
$P = \{S \rightarrow aSb, \;\; S \rightarrow aTb, \;\; T \rightarrow cTd, \;\; T \rightarrow cd\}$.

iv. Let $M = \{S, I. \delta, S_0, F\}$ be a Non-Deterministic Finite Automata (NDFA).
Where S is a finite set of states, I is a finite set of input symbols, δ is the transition function, A is the initial state, and F is the set of final states.
Transition Table for the above Non-Deterministic Finite Automata as follows:

States	Inputs		
	a	**b**	**c**
A	{B, E}	{A}	-
B	{B}	{C}	{A}
C	-	{C}	{D}
D	{D}	{D}	
E	-	{C, F}	{A}
F	{F}	-	{F}

The initial state is **A**, and the set of final states is {**D**, **F**}.

a) Depict the finite automaton's transition graph. **[10%]**

b) Show that the string ***aacabca*** is accepted by the Non-deterministic Finite Automaton by applying the transition function. **[30%]**

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