

The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc. /B. Ed Degree Programme



Department	: Mathematics
Level	: 04
Name of the Examination	: Final Examination
Course Title and - Code	: Vector Calculus – ADU4302/ADE4302
Academic Year	: 2024/2025
Date	: 24.11.2024
Time	: 9.30 a.m. To 11.30 a.m.
Duration	: Two Hours.

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of (6) questions in (2) pages.
3. Answer any (4) questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Involvement in any activity that is considered as an exam offense will lead to punishment.
6. Use blue or black ink to answer the questions.
7. Clearly state your index number in your answer script.

1. (a) State and sketch the domain of the function $f(x, y) = \sqrt{25 - x^2 - y^2}$.

(b) Sketch at least three level curves of the function $f(x, y) = \sqrt{25 - x^2 - y^2}$.

(c) Find the following limits if they exist justifying your answer.

(i) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^2}{x^4 + y^2},$

(ii) $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^2}.$

(d) Discuss the continuity of the following function at $(0, 0)$.

$$f(x, y) = \begin{cases} \frac{4xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(You may use your conclusion regarding c(ii).)

2. (a) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D . Explain briefly how you could determine its nature.

(b) Find the maximum and minimum values of the function $f(x, y) = x^3 + 3xy^2 - 15x + y^3 - 15y$ and determine their nature.

(c) The electrical potential (voltage) in a certain region of space is given by the function $V(x, y, z) = x^3 - xy^2 - z$.

(i) Find the rate of change of the voltage at point $(1, 1, 0)$ in the direction of the vector $\underline{v} = 2\underline{i} - 3\underline{j} + 6\underline{k}$.

(ii) In which direction does the voltage change most rapidly at point $(1, 1, 0)$?

(iii) What is the maximum rate of change of the voltage at point $(1, 1, 0)$?

3. (a) Prove that $\text{grad } \phi$ is a vector normal to the contour surface $\phi(x, y, z) = c$, where c is a constant.

(b) (i) Show that the equation of the tangent plane and normal line to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is given by

$$(x - x_0) \left(\frac{\partial F}{\partial x} \right)_P + (y - y_0) \left(\frac{\partial F}{\partial y} \right)_P + (z - z_0) \left(\frac{\partial F}{\partial z} \right)_P = 0 \text{ and}$$

$$\frac{(x-x_0)}{\left(\frac{\partial F}{\partial x}\right)_P} = \frac{(y-y_0)}{\left(\frac{\partial F}{\partial y}\right)_P} = \frac{(z-z_0)}{\left(\frac{\partial F}{\partial z}\right)_P}.$$

(ii) Using the above result, find the equation of the tangent plane and normal line to the surface $x^2 + 2xy - y^2 + z^2 = 7$ at the point $P(1, -1, 3)$.

(c) Prove that the vector field $\underline{F} = (e^x \cos y + yz)\underline{i} + (xz - e^x \sin y)\underline{j} + (xy + z)\underline{k}$ is conservative. Find the corresponding scalar potential function ϕ such that $\underline{F} = \nabla\phi$.

4. (a) State Gauss' Divergence Theorem.

(b) Verify the above theorem considering the vector field $\underline{F} = 4x\underline{i} - 2y^2\underline{j} + z^2\underline{k}$ defined over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$ and $z = 3$ planes.

5. (a) State Stokes' Theorem.

(b) Verify the above theorem considering the vector field $\underline{F} = y\underline{i} - x\underline{j}$ where S is the surface of the hemisphere $S: x^2 + y^2 + z^2 = 9$ and $z \geq 0$, C is the perimeter curve $x^2 + y^2 = 9$.

6. (a) Suppose that S is a plane surface lying in the xy -plane and bounded by a closed curve C . If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$.

(b) Verify the above result for the integral $\oint_C \{(6y + x)dx + (y + 2x)dy\}$, where C is the path along the curve $(x - 2)^2 + (y - 3)^2 = 4$ and oriented in the counterclockwise direction.