The Open University of Sri Lanka
B.Sc./B. Ed Degree Programme
Final Examination – 2024/2025
Pure Mathematics – Level 04
PEU4315 – Continuous Functions
Duration: Two hours.

Date: 11.12.2024 **Time: 09.30 a.m. – 11.30 a.m.**

Answer **Four** Questions only.

1. (a). State the definition of a limit point.

   (b). Let $f(x) = x^3$, for all $x \in \mathbb{R}$.

   Prove that $\lim_{x\to 2} f(x) \neq 4$.

   (c). Prove that the set of natural numbers, $\mathbb{N}$ has no limit points.

2. (a). Suppose $E \subseteq \mathbb{R}$, c is a limit point of E, $f, g: E \to \mathbb{R}$ are functions such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Prove that $\lim_{x\to c} f(x)\, g(x)$ exists and $\lim_{x\to c} f(x)\, g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.

   (b). Let $E = \mathbb{Q} \cap [0,2]$ and $f: E \to \mathbb{R}$ be $f(x) = (x^2 + 3)\sqrt{x+7}, x \in E$. Find $\lim_{x\to 2} f(x)$.

3. (a). By using the Sandwich theorem, prove that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

   (b). Prove that if $\lim_{x\to\infty} f(x) = l$ exists for some $l \in \mathbb{R}$, then l is unique.

   (c). Let $f(x) = \frac{1}{x}$, for each $x \in (0, \infty)$. Prove that f is unbounded and $\lim_{x\to\infty} f(x) = 0$.

4. (a). Let f, g be functions, c_1, c_2 be real numbers such that $(c_1, \infty) \subseteq Domn(f)$ and $(c_2, \infty) \subseteq Domn(g)$. Suppose $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist. Prove that $\lim_{x\to\infty} [f(x)g(x)]$ exists and $\lim_{x\to\infty} [f(x)g(x)] = \lim_{x\to\infty} f(x) \lim_{x\to\infty} g(x)$.

   (b). Let $f(x) = \frac{1}{(x-3)^2}, x \in \mathbb{R} \setminus \{3\}$. Show that $\lim_{x\to 3} f(x) = \infty$.

5. (a). (i). Suppose f is a function defined on an interval (a, b) and f is continuous at c, where $c \in (a, b)$. Prove that $|f(x)|$ is continuous at c.

   (ii). Is it true that $|f(x)|$ is always discontinuous at a point where the function $f(x)$ is discontinuous at the same point? Justify your answer.

   (b). Let $f(x) = \frac{x+1}{x^2+2}, x \in \mathbb{R}$.

   Using definition, prove that $f(x)$ is continuous at 3.

6. (a). Define $h: (-1,\ 5) \to \mathbb{R}$ by $h(x) = \sqrt{5 - x^2 + 4x}$ for each $x \in (-1, 5)$. Show that h is continuous on $(-1, 5)$.

   (b). Suppose f is a function defined on an interval I and f is uniformly continuous on I. Prove that f is continuous on I.

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