

The Open University of Sri Lanka

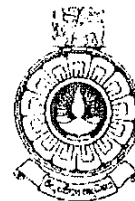
B.Sc/B.Ed. Degree Programme

Final Examination - 2024/2025

Pure Mathematics - Level 05

PEU5305 – Complex Analysis I

Duration: - Two hours



034

Date: - 10-12-2024

Time: - 09.30 a.m. – 11.30 a.m.

Answer **FOUR** Questions **ONLY**.

1.

- Determine where the function  $f(z) = 2x^3 + xy^2 + i\left(\frac{1}{3}y^3 + 6x^2y\right)$  is differentiable and where it is analytic.
- Find a harmonic conjugate  $v(x, y)$  of the harmonic function  $u(x, y) = x^2 - y^2$  and express  $u(x, y) + i v(x, y)$  as a function of  $z$ .
- Let  $f(z)$  be analytic in a region  $G$ . Show that if  $|f(z)|$  is constant in  $G$ , then  $f(z)$  is constant in  $G$ .

2.

- Prove that
  - $e^{i\theta} = \cos \theta + i \sin \theta$  for all  $\theta \in \mathbb{R}$ .
  - $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$  for all  $z_1, z_2 \in \mathbb{C}$ .
  - $\cosh^2 z - \sinh^2 z = 1$  for all  $z \in \mathbb{C}$ .
- Solve the equation  $e^{3z} = 1 + i$ .
- Find the values of  $\log(-1 - i)$  and  $\text{Log}(-1 - i)$ .

3.

a) State Cauchy's Integral Formula.

b) Using Cauchy's Integral Formula, evaluate the integral  $\int_C \frac{\cosh z}{z+i} dz$ , where  $C$  is the circle with radius 2 centered at  $2i$  oriented counterclockwise.

c) Apply Cauchy's integral formula to  $\sin z$  with a suitable contour to show that

$$\int_0^{2\pi} \sinh(\sin \theta) \cos(\cos \theta) d\theta = 0.$$

4.

a) State Green's Theorem in the plane.

b) Evaluate the line integral  $\int_C (x^2 - y^2) dx - 2xy dy$ , where  $C$  is the unit circle given by  $x = \cos \theta, y = \sin \theta; 0 \leq \theta \leq 2\pi$ .

i. Using Green's Theorem in the plane.

ii. Using the parameterization of  $C$ .

c) Let  $A$  be the area bounded by a simple closed contour  $C$ . Show that  $A = -\int_C y dx$ .

Using the above formula, find the area enclosed by the cycloid  $x = t - \sin t$  and  $y = \cos t - 1$ , for  $0 \leq t \leq 2\pi$ , and the  $x$ -axis.

5.

a) Find the radius of convergence of the power series  $\sum_{n=0}^{\infty} \left( \frac{2n+3}{3n+1} \right) (z-1)^n$ .

b) Find the Maclaurin series of  $f(z) = \sin^3 z$ .

c) Using Laurent's theorem show that  $\sin\left(z + \frac{1}{z}\right) = \sum_{n=0}^{\infty} C_n \left( z^{2n+1} + \frac{1}{z^{2n+1}} \right)$  for

$$0 < |z| < \infty \text{ where } C_n = \frac{1}{2\pi} \int_0^{2\pi} \sin(2 \cos \theta) \cos(2n+1)\theta d\theta \text{ for } n = 0, 1, 2, \dots$$

d) Find and classify the singularities of the function  $f(z) = \frac{e^{zz}}{(z-1)(z+i)^2}$ .

6.

a) State Cauchy's Residue Theorem.

b) Using Cauchy's Residue Theorem, calculate each of the following contour integrals:

i.  $\int_C \cot z \, dz$ , where  $C$  is the circle  $|z| = 4$ , oriented counterclockwise.

ii.  $\int_C \frac{3z}{z^2+9} \, dz$ , where  $C$  is the circle  $|z| = 3$ , oriented counterclockwise.

c) Using Cauchy's Residue Theorem, show that  $\int_0^{2\pi} \frac{1}{3+2\cos\theta} \, d\theta = \frac{2\sqrt{5}\pi}{5}$ .