The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2024/2025
Pure Mathematics - Level 05
PEU5305 – Complex Analysis I
Duration: - Two hours

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Date: - 10-12-2024 **Time: - 09.30 a.m. – 11.30 a.m.**

Answer **FOUR** Questions **ONLY**.

1.

a) Determine where the function $f(z) = 2x^3 + xy^2 + i\left(\frac{1}{3}y^3 + 6x^2y\right)$ is differentiable and where it is analytic.

b) Find a harmonic conjugate $v(x, y)$ of the harmonic function $u(x, y) = x^2 - y^2$ and express $u(x, y) + i\,v(x, y)$ as a function of z.

c) Let $f(z)$ be analytic in a region G. Show that if $|f(z)|$ is constant in G, then $f(z)$ is constant in G.

2.

a) Prove that

i. $e^{i\theta} = \cos\theta + i\sin\theta$ for all $\theta \in \mathbb{R}$.

ii. $e^{z_1+z_2} = e^{z_1}.e^{z_2}$ for all $z_1, z_2 \in \mathbb{C}$.

iii. $\cosh^2 z - \sinh^2 z = 1$ for all $z \in \mathbb{C}$.

b) Solve the equation $e^{3z} = 1 + i$.

c) Find the values of $\log(-1 - i)$ and $\mathrm{Log}(-1 - i)$.

3.

a) State Cauchy's Integral Formula.

b) Using Cauchy's Integral Formula, evaluate the integral $\int_C \frac{\cosh z}{z+i} dz$, where C is the circle with radius 2 centered at $2i$ oriented counterclockwise.

c) Apply Cauchy's integral formula to $\sin z$ with a suitable contour to show that $\int_0^{2\pi} \sinh(\sin\theta)\cos(\cos\theta)d\theta = 0.$

4.

a) State Green's Theorem in the plane.

b) Evaluate the line integral $\int_C (x^2 - y^2)dx - 2xy\,dy$, where C is the unit circle given by $x = \cos\theta, y = \sin\theta;\ 0 \le \theta \le 2\pi$.

i. Using Green's Theorem in the plane.

ii. Using the parameterization of C.

c) Let A be the area bounded by a simple closed contour C. Show that $A = -\int_C y\,dx$.

Using the above formula, find the area enclosed by the cycloid $x = t - \sin t$ and $y = \cos t - 1$, for $0 \le t \le 2\pi$, and the x-axis.

5.

a) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \left(\frac{2n+3}{3n+1}\right)(z-1)^n$.

b) Find the Maclaurin series of $f(z) = \sin^3 z$.

c) Using Laurent's theorem show that $\sin\left(z + \frac{1}{z}\right) = \sum_{n=0}^{\infty} C_n \left(z^{2n+1} + \frac{1}{z^{2n+1}}\right)$ for $0 < |z| < \infty$ where $C_n = \frac{1}{2\pi}\int_0^{2\pi} \sin(2\cos\theta)\cos(2n+1)\theta\ d\theta$ for $n = 0, 1, 2, \ldots$

d) Find and classify the singularities of the function $f(z) = \frac{e^{2z}}{(z-1)(z+i)^2}$.

6.

a) State Cauchy's Residue Theorem.

b) Using Cauchy's Residue Theorem, calculate each of the following contour integrals:

i. $\int_C \cot z \, dz$, where C is the circle $|z| = 4$, oriented counterclockwise.

ii. $\int_C \frac{3z}{z^2+9} \, dz$, where C is the circle $|z| = 3$, oriented counterclockwise.

c) Using Cauchy's Residue Theorem, show that $\int_0^{2\pi} \frac{1}{3+2\cos\theta} \, d\theta = \frac{2\sqrt{5}\pi}{5}$.

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