The Open University of Sri Lanka

B.Sc/B.Ed Degree Programme

Pure Mathematics – Level 04

PEU4301-Real Analysis II

Final Examintaion - 2024/2025

Date: 26.04.2024	**Time: 09.30 a.m. – 11.30 a.m.**

General Instructions

- This paper consists of **TWO** sections, Section **A** and Section **B**. Section **A** is compulsory and it consists of **FIVE** Structured Essay Questions and carries 100 marks.
- Section B consists of **FIVE** essay-type questions and answer only **THREE** of them. Each question in Section B carries 100 marks.
- This paper consists of 03 pages.

SECTION A

Answer all the questions in this section.

Let the function $g: \mathbb{R} \longrightarrow \mathbb{R}$ be given by $g(x) = \begin{cases} x^2, & x \in [0, \infty) \\ 1, & x \in (-2,0) \end{cases}$

Answer the questions (a) to (d) by using this function g.

(a) Find the $\lim_{x \to 1^-} g(x)$ by using the $\varepsilon - \delta$ definition.

(b) Is the function g left continuous at $x = 0$? Justify your answer.

(c) Is the function g continuous at $x = 0$? Justify your answer.

(d) Is the function g differentiable at $x = 0$? Justify your answer.

(e) Let the function f be defined as $f(x) = 2x + 1, x \in \mathbb{R}$.

Prove that the function f is differentiable at $x = 1$.

SECTION B

1. (a) Let $f(x) = \frac{1}{x}, x > 0$.
   (i) Show that $\lim_{x \to \infty} f(x) = 0$.
   (ii) Suppose you need to find the $\lim_{x \to 0^+} \frac{1}{x}$. What are the changes you need to include in the $\varepsilon - \delta$ definition used in part (i).

   (b)
   (i) Find two functions $g(x)$ and $h(x)$ which satisfy the inequality
   $g(x) < r(x) < h(x)$ for $x \geq 0$, where $r(x) = \frac{1}{x^2+1}, x \geq 0$.
   (ii) Hence determine $\lim_{x \to \infty} r(x)$.

   (c) Prove that $\lim_{x \to 1^+} f(x) = \infty$, where $f(x) = \frac{1}{x-1}, x \neq 1$.

2. (a) Let a, b, c be real numbers and f be a function such that $a < b < c$ and $(a, c) \subseteq$ Domn(f).
   Prove that if f is left-continuous at b and f is right-continuous at b then f is continuous at b.

   (b) Let $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ -1, & x = 1. \end{cases}$

   Prove or disprove each of the following.
   (i) f is continuous on $[0,1)$
   (ii) f is not continuous on $[0,1]$
   (iii) f^2 is continuous on $[0,1]$.

   (c) Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ such that f is nowhere continuous and f^2 is everywhere continuous on $\mathbb{R}$.

3. (a) Let $g(x) = 1 - x, x \in (0,1)$.
   Prove that $\lim_{x \to 1^-} g(x) = 0$.

   (b) Let $h(x) = (1 - x)\sin\frac{1}{(1-x)}, x \in (0,1)$
   Determine the limit $\lim_{x \to 1^-} h(x)$ using the Sandwich theorem.

   (c) Does the existence of $\lim_{x \to c^-} f(x)g(x)$ imply the existence of $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^-} g(x)$?
   Justify your answer.

(d) Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{Q}^c \end{cases}$.

Prove that $\lim_{x \to 0} f(x)$ does not exist.

4. (a) Let f be a function and c is a real number such that f is differentiable at c.

Prove that f is continuous at c.

(b) Let $f(x) = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$.

(i) Show that the function f continuous everywhere.

(ii) Is the function f differentiable everywhere? Justify your answer.

(c) Give an example of a function that is not continuous everywhere.
Hence, discuss whether it is differentiable everywhere.

5. (a) Prove that $\lim_{x \to -\infty} f(x) = \infty$, where $f(x)=\frac{x^6+2}{x^4+2}$, $x \in \mathbb{R}$.

(b) Let f be a function such that $\lim_{x \to -\infty} f(x) = \infty$.

Prove that $\lim_{x \to -\infty} \alpha f(x) = \begin{cases} -\infty, & \alpha > 0 \\ 0, & \alpha = 0 \\ \infty, & \alpha < 0. \end{cases}$

(c) Find two functions f and g such that $\lim_{x \to -\infty} f(x) = \infty$, $\lim_{x \to -\infty} g(x) = -\infty$, and $\lim_{x \to -\infty} [f(x) + g(x)] = 2025$,