The Open University of Sri Lanka

B.Sc/B.Ed. DEGREE, CONTINUING EDUCATION PROGRAMME

Final Examination 2023/2024

Level 03 Pure Mathematics

PEU3202/PEE3202 Vector Spaces

Duration: - Two Hours

Date: - 02-04-2024 **Time: 9.30 a.m. to 11.30 a.m.**

Answer four questions only

1.

(a) Suppose V is a vector space over a field F. Prove that

(i) $0 \cdot x = 0$ for all $x \in V$ and $0 \in F$.

(ii) $\alpha \cdot 0 = 0$ for all $\alpha \in F$ and $0 \in V$.

(iii) $(-\alpha)x = -(\alpha x)$ for all $\alpha \in F$ and $x \in V$.

(b) Let $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$. For every $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M$,

define $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$ and $\alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ 2\alpha c_1 & 3\alpha d_1 \end{bmatrix}$

for $\alpha \in \mathbb{R}$, where $\mathbb{R}$ is the real number field. Is M a vector space over the field of real numbers under these operations? Justify your answer.

(c) Determine whether the three vectors $(1, 2, 2), (1, -1, 2)$ $(1, 0, 1)$ are lineally independent over the usual vector space $\mathbb{R}^3$ over $\mathbb{R}$.

2.

(a) Let V be a vector space over the field F and $W \subseteq V$ and $W \neq \phi$. Show that W *is* a subspace of a vector space of V over F if and only if for all $\alpha, \beta \in F$ and $x, y \in W$, $\alpha x + \beta y \in W$.

(b) Determine whether the following sets are subspaces of the vector space $\mathbb{R}^4$ over the field $\mathbb{R}$ under the usual addition and scalar multiplication. In each case justify your answer.

(i) $A = \{(a,b,c,d) \mid a,b,c,d \in \mathbb{R};\ b = 2a + a^2 \text{ and } c = d\}$

(ii) $B = \{(a,b,c,d) \mid a,b,c,d \in \mathbb{R};\ a + b = c + d\}$

(c) If α, β and γ are linearly independent vectors in V over a field F, prove that $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$ are also linearly independent.

3.

(a) Suppose V is a vector space over the field F. Show that if $\beta \in V$ is a linear combination of the set of vectors $\alpha_1, \alpha_2, \cdots, \alpha_n \in V$, then the set $\{\beta, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ is linearly dependent.

(b) Suppose W is a subspace of a finite dimensional vector space V over the field F, then prove that $\dim W \leq \dim V$

(c) Let $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations of vector spaces over the same field F. Prove that the composition $T_2 \circ T_1 : U \to W$ is also a linear transformation.

4.

(a) Let $T : V \to W$ be a linear transformation where V and W are vector space over the field F. Show that

(i) $T(0) = 0$

(ii) $KerT = \{0\}$ if and only if T is one to one.

(b) Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$. Note that V and W are vector spaces over the field $\mathbb{R}$ under the usual addition and scalar multiplication.

Consider the mapping $T : V \to W$ defined by $T(x,y,z) = (x + y, x + 2z)$.

(i) Show that T is a linear transformation.

(ii) Find the Kernel of T.

(iii) Is T an Isomorphism? Justify your answer.

5.

(a) Let $M = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a,b,c,d \in \mathbb{R} \}$. Note that M is a vector space over the field $\mathbb{R}$ under the usual matrix addition and scalar multiplication.

Let the mapping $T : M \to M$ be defined by $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a+b & b \\ 3c & d \end{bmatrix}$. Note that T is a linear Transformation

Determine whether the following sets are invariant subspaces of the vector space M over the field $\mathbb{R}$ under T.

(i) $W = \left\{ \begin{bmatrix} a & b \\ a+b & 0 \end{bmatrix} \middle| a,b \in \mathbb{R} \right\}$

(ii) $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a,b,c \in \mathbb{R} \right\}$

(b)

(i) Define an inner product space.

(ii) Let V be an inner product space over a field F. Prove that for $x_1, x_2, y_1, y_2 \in V$,

$$< x_1 + x_2, y_1 + y_2 > = < x_1, y_1 > + < x_1, y_2 > + < x_2, y_1 > + < x_2, y_2 >$$

(iii) Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$ where $u, v \in \mathbb{R}^3$.

Define $< u, v > = {x_1}^2{y_1}^2 + {x_2}^2{y_2}^2 + x_3y_3$. Is $< u, v >$ an inner product on $\mathbb{R}^3$? Justify your answer.

6.

(a) Let U be a subspace of a vector space V over a field F, $T : U \to V$ be a linear transformation and Ker T $= \{0\}$, Let $S = \{ u_1, u_2, \dots u_n\}$ be a linearly independent set of vectors in U. Is the set $T(S) = \{ T(u_i) | u_i \in S\}$ linearly independent? Justify your answer.

(b) Let $W = \{(1, 1, -2, 0), (2, 1, -3, 0), (-1, 0, 1, 0), (0, 1, -1, 0)\}$. Find a basis for the subspace $Sp{<}W{>}$ of $\mathbb{R}^4$ over the field $\mathbb{R}$

(c) Show that the three vectors $u_1 = (1, 2, 2)$, $u_2 = (1, -1, 2)$ and $u_3 = (1, 0, 1)$ form a basis for E^3, the usual Euclidean three space. Construct an orthonormal basis for E^3 out of $\{u_1, u_2, u_3\}$ using the Gram–Schmidt process.