The Open University of Sri Lanka
B.Sc. Degree Programme – Level 05
Closed Book Test (CBT) - 2009/2010
Pure Mathematics / Computer Science
PMU 3294/PME 5294/CSU 3276 – Discrete Mathematics

Model Answer

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01. (a) Conditional probability:

Let A and B be two events in a probability space (S, P). Then the **conditional probability** of the event A given that the event B has occurred is defined to be $P\left(A/B\right)$, where $P\left(A/B\right)=\dfrac{P(A\cap B)}{P(B)}$.

(b) Let (S, P) be a probability space and suppose that A_1, A_2, A_3 be three events in (S, P). Then

$$
\begin{aligned}
P(A_1\cap A_2\cap A_3) &= P((A_1\cap A_2)\cap A_3)\\
&= P(A_1\cap A_2)\,P\left(A_3/A_1\cap A_2\right)\\
&= P(A_1)\,P\left(A_2/A_1\right)P\left(A_3/A_1\cap A_2\right).
\end{aligned}
$$

(c) Sample space $S=\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$

(i) Let A: "They have more than one boy"

$$A=\{BBB, BBG, BGB, GBB\} \quad\Rightarrow\quad P(A)=4/8$$

Let B: "They have at least one boy"

$$B=\{BBB, BBG, BGB, BGG, GBB, GBG, GGB\} \quad\Rightarrow\quad P(B)=7/8$$

Also $P(A\cap B)=P(A)=4/8$

$$\therefore P\left(A/B\right)=\frac{P(A\cap B)}{P(B)}=\frac{4/8}{7/8}=\frac{4}{7}$$

(ii) Let C: "The first child is a girl"

$$C=\{GBB,GBG,GGB,GGG\} \Rightarrow P(C)=4/8$$

$$A\cap C=\{GBB\} \Rightarrow P\left(A/C\right)=\frac{P(A\cap C)}{P(C)}=\frac{1/8}{4/8}=\frac{1}{4}$$

(iii) Since $P(A)=\frac{4}{8}$, $P(C)=\frac{4}{8}$ and $P(A\cap C)=\frac{1}{8}$

$$P(C)P(A)=\frac{4}{8}\times\frac{4}{8}=\frac{1}{4}\neq P(A\cap C)$$

$\therefore A$ and C are not independent.

02. (a) Degree of a vertex:

Let G be a graph (or a multi-graph) and suppose that v is a vertex of G. Then the **degree** of v, denoted by $\delta(v)$, is the number of edges of G which are incident on v or, in other words, the number of edges of G having v as an end point.

(b) Let $A=\left[a_{ij}\right]_{n\times n}$ be the adjacency matrix of G. Then the sum of all the entries in A is given by $\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}$.

We know that, if (v_i,v_j) is an edge of G. Then $a_{ij}=a_{ji}=1 \Rightarrow a_{ij}+a_{ji}=2$.

Thus each edge contributes 2 units to the sum.

Therefore $\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}=2|E(G)|$ ....(1)

By the definition of the degree of vertex v_i, $\delta(v_i)=\sum_{j=1}^{n}a_{ij}$ .... (2)

By (1) and (2), $\sum_{i=1}^{n}\delta(v_i)=2|E(G)|$.

(c) $V(G)=\{v_1, v_2, v_3, v_4\}$, $E(G)=\{v_1 v_2, v_1 v_3, v_2 v_3, v_2 v_4, v_3 v_4\}$

(i) $\delta(v_1)=2$, $\delta(v_2)=3$, $\delta(v_3)=3$, $\delta(v_4)=2$

$$\sum_{i=1}^{4}\delta(v_i)=10=2\times 5=2\times(\text{number of edges of } G)$$

(ii) $A=\begin{bmatrix}0&1&1&0\\1&0&1&1\\1&1&0&1\\0&1&1&0\end{bmatrix}$

(iii) $A^2=\begin{bmatrix}2&1&1&2\\1&3&2&1\\1&2&3&1\\2&1&1&2\end{bmatrix}$ and $A^3=\begin{bmatrix}2&5&5&\boxed{2}\\5&4&5&5\\5&5&4&5\\\boxed{2}&5&5&2\end{bmatrix}$

Number of paths of length 3 joining v_1 and v_4 is 2, which are

$v_1 \to v_2 \to v_3 \to v_4$ and $v_1 \to v_3 \to v_2 \to v_4$.

(iv) $A+A^2+A^3=\begin{bmatrix}4&7&7&4\\7&7&8&7\\7&8&7&7\\4&7&7&4\end{bmatrix}$.

Notice that all the entries are non- zero, therefore G is connected

(v) No! Because G has cycles (for example : $v_1 \to v_2 \to v_3 \to v_1$)

03. (a) $f(n+2)+a\,f(n+1)+b\,f(n)=0$; where a,b are constants, is a second order difference equation with constant coefficients.

(b) If $g(n)$ is a solution of the equation in part (a), then

$g(n+2)+a\,g(n+1)+b\,g(n)=0$ .... (1)

If $h(n)$ is a solution of the equation in part (a), then

$h(n+2)+a\,h(n+1)+b\,h(n)=0$ .... (2)

Substitute $f(n)=\alpha\,g(n)+\beta\,h(n)$ to the equation in part (a). Then

$$\begin{aligned}
&\left(\alpha g(n+2)+\beta h(n+2)\right)+a\left(\alpha g(n+1)+\beta h(n+1)\right)+b\left(\alpha g(n)+\beta h(n)\right)\\
&=\alpha g(n+2)+a\alpha g(n+1)+b\alpha g(n)+\beta h(n+2)+a\,\beta h(n+1)+b\,\beta h(n)\\
&=\alpha\left[g(n+2)+a\,g(n+1)+b\,g(n)\right]+\beta\left[h(n+2)+a\,h(n+1)+b\,h(n)\right]\\
&=\alpha.0+\beta.0 \qquad (\textit{by } (1) \textit{ and } (2))\\
&=0
\end{aligned}$$

Therefore $\alpha\,g(n)+\beta\,h(n)$ is also a solution.

(c) (i) $2f(n+3)+3f(n+2)-f(n)=0$

Try: $f(n)=A^n$. Then the characteristic equation is $2A^3+3A^2-1=0$.

The solutions of the characteristic equation are $A=\frac{1}{2},\,-1,\,-1$

Therefore the general solution of the above difference equation is:

$$\alpha\left(\tfrac{1}{2}\right)^n+\beta(-1)^n+\gamma\,n(-1)^n$$

(ii) $f(n+2)-6f(n+1)+13f(n)=0$

Try: $f(n)=A^n$. Then the characteristic equation is $A^2-6A+13=0$.

The solutions of the characteristic equation are $A=3\pm 2i$

Therefore the general solution of the above difference equation is:

$$\rho^n\left(\alpha\sin n\theta+\beta\cos n\theta\right)$$

where $\rho=\sqrt{3^2+2^2}=\sqrt{13}$ and $\theta=\tan^{-1}\left(\tfrac{2}{3}\right)$