The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme-Level 05
Final Examination-2009/2010
Pure Mathematics
PMU3292/PME5292-Group Theory & Transformation-Paper I

Duration:- Two and Half Hours

Date:- 08.01.2010 **Time:-01.00 p.m.-03.30 p.m.**

Answer Four Questions only.

01. (a) Prove that the set of matrices $K=\left\{\begin{pmatrix} 2x & -2x \\ -2x & 2x \end{pmatrix} : x\in\mathbb{R}^*\right\}$ is a group under matrix multiplication, giving its identity and the inverse of a typical element.

(b) Let $S=\{(x,y): x\in\mathbb{R}, y\in\mathbb{R}^*\}$. An operation $\circ$ is defined on S by $(x_1,y_1)\circ(x_2,y_2)=(x_1y_2+y_1x_2, y_1y_2)$. Prove that $(S,\circ)$ is a group.

02. (a) Prove that the intersection of two subgroups H and K of a group G is a subgroup.
(b) Let $D=\mathbb{Z}\times\mathbb{Z}$ be a group under $*$ defined by $(a,b)*(c,d)=\left(a+c,(-1)^c b+d\right)$. Determine whether $H=\{(a,0)/a\in\mathbb{Z}\}$ and $K=\{(0,a)/a\in\mathbb{Z}\}$ are subgroups of D, where $\mathbb{Z}$ is the set of integers.

03. (a) Is $\mathbb{Q}^*$ a subgroup of the group of real numbers of the form $a+b\sqrt{2}$, where $a,b\in\mathbb{Q}$ and a, b not simultaneously zero, under the multiplication?

(b)Let $A_4 = \{i, \sigma_2, \sigma_5, \sigma_8, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\}$ where

$$i = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \sigma_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\sigma_8 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

$$\gamma_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \quad \gamma_7 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad \gamma_8 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Show that the $S = \{i, \sigma_2, \sigma_5, \sigma_8\}$ is a subgroup of A_4.

04. (a) Let $G = \mathbb{Z}$, the additive group of integers. What is $\langle 1 \rangle$?

(b) Find the subgroup of the multiplicative group of rationals generated by $\{2\}$.

(c) Determine the subgroup H of S_3 generated by the cycles $(1 \;\; 2 \;\; 3)$ and $(2 \;\; 3)$.

05. (a) If a and b are elements of a group G and $a^{-1}b^2a = ba$, then prove that $b = a$.

(b) Write $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 \end{pmatrix}$ as the product of disjoint cycles.

06. (a) Find the order of each of the elements 2,3,5 in the group $(\mathbb{Z}_7^*, \otimes)$,where $\otimes$ is defined by multiplication under non-zero integer modulo 7 . Show that every element of this group can be written as a power of the element 3.

(b) Find the order of each of the following matrices in the group $GL(2,\mathbb{R})$, the multiplicative group of 2×2 non-singular matrices over $\mathbb{R}$:

$(a)\, A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $\qquad (b)\, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$(c)\, C = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$