THE OPEN UNIVERSITY OF SRI LANKA
Faculty of Engineering Technology
Department of Mathematics & Philosophy of Engineering

Bachelor of Technology Honors in Engineering / Bachelor of Software Engineering Honors

Final Examination (2022/2023)
MHZ5355 / MHZ5375: Discrete Mathematics

Date: 22nd February 2024 (Thursday) **Time: 09:30 – 12:30**

Instructions:

- **This paper has four (4) sections, each with two (2) questions and a length of six (6) pages.**
- **Answer five (5) questions in total, including at least one from each section.**
- **All symbols are in standard notation and state any assumptions you made.**

SECTION – A

Q1.

i. Prove the following statements using Mathematical induction.

a) $7^n - 1$ is divisible by 6 for all integers $n \geq 1$. **[20%]**

b) $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \ldots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$ for all integers $n \geq 2$. **[20%]**

ii. Let a, b, c, and d be any integers. Prove that,

a) if $a|b$ and $a|c$, and $m, n \in \mathbb{Z}$, then $a|mb + nc$. **[10%]**

b) If $a|b$ and $c|d$, then $ac|bd$. **[10%]**

iii. Prove that if x is even, then $x^2 + 2x + 4$ is divisible by 4. **[15%]**
[Hint: use the results of part ii. a) and b)]

iv.

a) Find the $gcd(172, 20)$, and find the integers x and y such that $gcd(172,\ 20) = 172x + 20y$ by using the Euclidean Algorithm. **[15%]**

b) Determine all integer solutions of the following Diophantine equation:

$$172x_0 + 20y_0 = 1000.$$ **[10%]**

Q2.

i. Let n be a positive integer. For all $a, b, c \in \mathbb{Z}$

a) Prove that $a \equiv a \ (mod\ n)$. **[05%]**

b) If $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$, then show that $a + c \equiv b + d \ (mod\ n)$ and $ac \equiv bd \ (mod\ n)$. **[30%]**

ii.

a) Find all solutions to $3x \equiv 1 \ (mod\ 6)$, between 0 and 5 inclusive. **[10%]**

b) Find the all incongruent solutions of $6x \equiv 9 \ (mod\ 15)$. **[15%]**

iii. Solve the following system of congruence using the Chinese Remainder Theorem.

$$x \equiv 3 \ (mod\ 5)$$
$$x \equiv 1 \ (mod\ 7)$$
$$x \equiv 6 \ (mod\ 8).$$

[40%]

SECTION – B

Q3.

i. Determine whether the operation " $*$ " is binary or not on the sets given below. If " $*$ " is a binary operation, prove your answer; otherwise, if it is not, justify it with a counter example. **[30%]**

a) $x * y = x^y + y^x$; $\quad x, y \in \mathbb{Z}$,

b) $x * y = x.|y|$; $\quad x, y \in \mathbb{R}$,

c) $x * y = 2x - y$; $\quad x, y \in \mathbb{N}$.

ii. Consider the binary operation " $*$ " defined on the set $A = \{a, b, c, d\}$ by the following table:

$*$	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative? Justify your answer. **[25%]**

iii. Define an abelian group $(G, *)$ in usual notation.
Let $S = \mathbb{R}\backslash\{-1\}$ and define a binary operation on S by $a * b = a + b + ab$.
Prove that $(S,*)$ is an abelian group. **[45%]**

Q4.

i. Define a semi-group $(G, \#)$ in usual notation.
Let operations "#" : $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as follows.

a) "$a \# b = a^b$" for all $a, b \in \mathbb{R}$;
b) "$c \# d = 3(c + d)$" for all $c, d \in \mathbb{R}$;
c) "$e \# f = \sqrt{ef}$" for all $e, f \in \mathbb{R}^+$.

Verify that where $(\mathbb{R}, \#)$ a semi-group or not for each of the above cases. **[45%]**

ii. Let $R=\{0, 1, 2, 3, 4, 5, 6\}$ be a group under the operation $\oplus_7$. The operation $\oplus_7$ is defined by $a \oplus_7 b=r$ and $0\leq r \leq 6$, where r is the non-negative remainder when ordinary addition $a+b$ is divided by 7. **[25%]**

a) Determine the identity element of R.
b) Determine the inverse of each element $a\in R$.

iii. Define a homomorphism for group in usual notation. **[30%]**
Let G_1 be the additive group of real numbers, $(\mathbb{R}, +)$ and G_2 be the multiplicative group of positive real numbers, $(\mathbb{R}^+, .)$ and $f: G_1 \to G_2$ defined by $f(x) = e^x$, for any $x \in \mathbb{R}$.

a) Show that f is a homomorphism.
b) Is f an isomorphism? Justify your answer.

SECTION – C

Q5.

i. For the following cases, express graphs using diagrammatic representation.
Determine whether they are a simple graph, simple digraph, or multi-graph.

a) $H_1 = \{V_1, E_1\}$ where $V_1 = \{-2, -1,0,1,2\}$ and $E_1 = \{(x, y),\ x = y + 1\}$. **[10%]**
b) $H_2 = \{V_2, E_2\}$ where $V_2 = \{ 1,2,3,4\}$ and $E_2 = \{(x, y),\ y = x^2\}$. **[10%]**
c) $H_3 = \{V_3, E_3\}$ where $V_3 = \{1, 2, 3, 4, 5\}$ and $E_3 = \{(x, y),\ x + y \ is \ odd\}$. **[10%]**

ii. A graph has 26 vertices and 58 edges. There are five vertices of degree 4, six vertices of degree 5, and seven vertices of degree 6. If the remaining vertices all have the same degree, what is their degree? **[15%]**

iii. G is the graph whose adjacency matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

a) Without drawing the graph of G, explain whether G is connected or not. **[15%]**

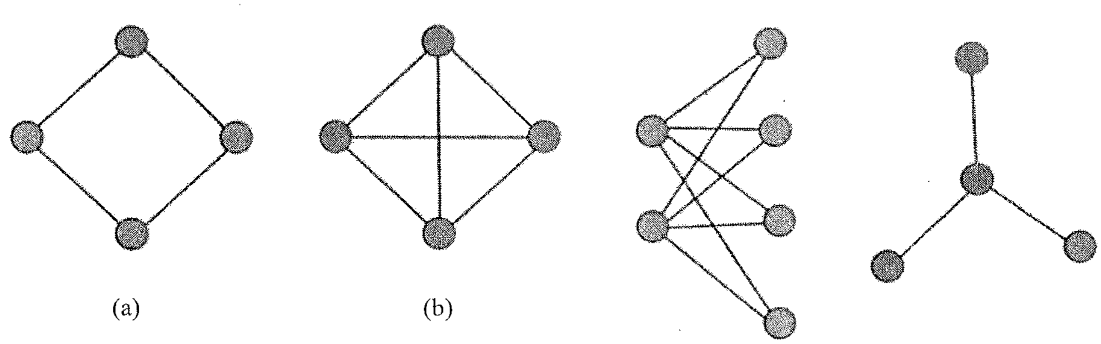
b) If $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, then find the number of different paths (walks) of length four (4) joining vertices v_5 and v_5. **[10%]**

c) Draw the graph of an adjacency matrix A. **[05%]**

iv. A graph has 24 vertices and 30 edges. It has five vertices of degree 4, seven vertices of degree 1, and seven vertices of degree 2. All other vertices have degree 3 or 4. How many vertices of degree 4 are there? **[25%]**

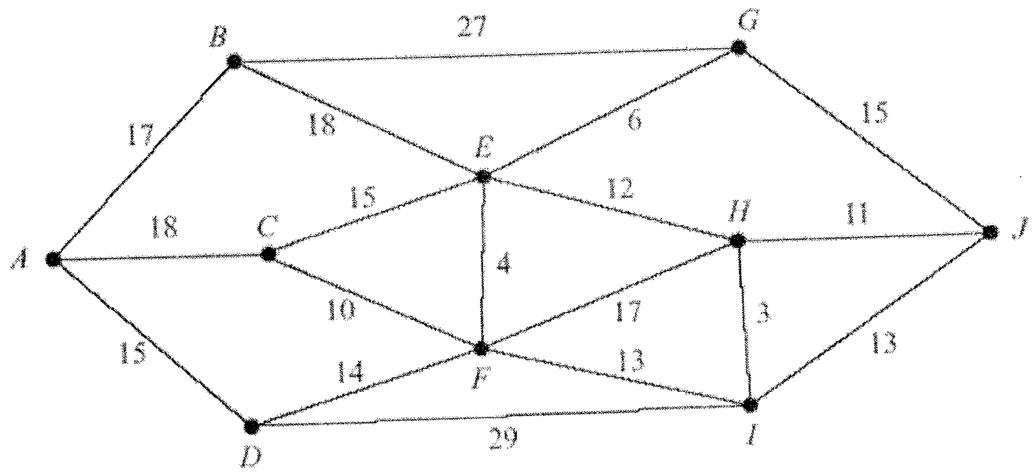
Q6.

i. Determine whether the graphs shown below are Eulerian or Hamiltonian. Justify your answers. **[20%]**



(a) (b) (c) (d)

ii. Use Dijkstra's algorithm to find the shortest route between node A and every other node in the following network. **[60%]**



iii. Draw a tree for the following case: **[20%]**

"Twelve vertices exactly four of which has degree two".

SECTION – D

Q7.

i. Iterate the Eco-system growth given by the relation $t_{n+1} - \lambda t_n = 0$ starting with $t_0 = 0.3$, taking $\lambda = 0.5$, and draw the diagram. Hence deduce t_n as $n \to \infty$. (At least 5 iteration steps are necessary) **[20%]**

ii. Consider the one-dimensional self-feedback system represented by $x_{n+1} = 2\,x_n(1 - x_n)$, with $x_0 = 0.4$. Using 5 iterations, determine the convergent value of the system. **[20%]**

iii. Consider the iterations given by $Z_{n+1} = Z_n^2$ and suppose that $Z_0 = (1 + i)$. Find Z_1 and up to Z_4. Discuss the solution Z_n as $n \to \infty$. **[10%]**

iv. Suppose a system with two unknowns x and y, is modeled in the form of a system of differential equations

$$\frac{dx}{dt} = -5x + 2y$$
$$\frac{dy}{dt} = -9x + 6y$$

with initial conditions $x = 3$ and $y = 10$ when $t = 0$.
Find the space values $(x_t,\ y_t)$, for $t = 0, 1, 2$. **[50%]**

Q8.

i. Let $L = \{1, 15, 51\}$ and $M = \{51, 515, 151\}$ be languages. Find the concatenations LM and ML. **[20%]**

ii. Suppose $L(G) = \{a^m b^n \mid m > 0 \text{ } and \text{ } n \geq 0\}$. Find out the grammar G which produces L. **[20%]**

iii. Show that the string $\{(x + y) * x - z * y/(x + x)\}$ is a sentence generated by the grammar and starting symbol S and production P. **[20%]**
$P = \{S \rightarrow x, S \rightarrow y, S \rightarrow z, S \rightarrow S + S, S \rightarrow S - S, S \rightarrow S * S, S \rightarrow S/S, S \rightarrow (S)\}$.

iv. Let $M = \{S, I, \delta, S_0, F\}$ be a Non-Deterministic Finite Automata (NDFA).
Where S is a finite set of states, I is a finite set of input symbols, δ is the transition function, S_0 is the initial state, and F is the set of final states.
Transition Table for the above Non-Deterministic Finite Automata as follows:

States	Inputs	
	0	**1**
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	-	$\{q_2\}$
q_2	$\{q_3\}$	-
q_3	-	$\{q_4\}$
q_4	$\{q_4\}$	$\{q_4\}$

The initial state is $\boldsymbol{q_0}$, and the set of final states is $\{\boldsymbol{q_4}\}$.

a) Depict the finite automaton's transition graph. **[10%]**
b) Show that the string **1011011** is accepted by the Non-deterministic Finite Automaton by applying the transition function. **[30%]**

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