**MPZ 4230 – Engineering Mathematics II**
**Assignment No. 01 – Academic Year 2006**

1. (a). Let $\underline{F}(x, y, z) = yz\,\underline{i} + (y^2 + xz)\,j + xy\,\underline{k}$
Is this vector field conservative?
If so find $\varnothing$ such that $\underline{F} = \underline{\nabla}\phi$

(b). Find the equations of the tangent plane and normal line to the surface $x^2yz + 3y^2 = 2xz^2 - 8z$ at the point $(1, 2, -1)$

2. Let $f(x, y, z) = x \sin(yz)$
(i). Find $\nabla f$
(ii). The directional derivative of f at (1, 3, 0) in the direction $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$

3. Suppose that the temperature of a point (x, y, z) in space is given by
$T(x, y, z) = \dfrac{80}{(1 + x^2 + 2y^2 + 3z^2)}$, where T is measure in °C and x, y, z in meters. In which direction does the temperature increase test of the point (1, 1, -2)? What is the maximum rate of increase ?

4. Let $f(x, y) = 2x^3 - 6x^2y + 18y^3 + 3x^2$.
(i). Find and classify the critical points of f(x, y)
(ii). $0 \le a \le \frac{3}{4}$, & let T be the triangle with vertices (o, o) (a, o) (a, a). Let R be the set of all points inside on
Find a such that max $(x, y) \in R$ $f(x, y = 5a^2)$

5. Prove that cauchy's integral formula.
Evaluate $\dfrac{1}{2zi}\oint_c \dfrac{e^z}{z-2}\,dz$
(i). If C is $|z| = 1$
(ii). If C is $|z| = 3$



6. (a). Find the taylor series for the following functions of the indicated points.
(i). $e^z$ at $z = 0$
(ii). $\dfrac{1-z}{z-3}$ centered at z = 1

(b). Show that $\ln(1 + z) = \sum_{n=1}^{\alpha} \dfrac{(-1)^n z^n}{n}$

Please send your answers according to the Activity Diary Due Date to the following address

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## Model Answer 01 – MPZ 4230
## Academic Year 2006

1. Curl $\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & (y^2+xz) & xy \end{vmatrix}$

$= \left|(x-x)\underline{i}(y-y)\underline{j}+(z-z)\underline{k}\right|$

$= 0$

$\therefore$ $\underline{F}$ is conservative

Therefore then exist $\phi$ such that $F = \underline{\nabla}\phi$

$yz\underline{i} + (y^2+xz)\,\underline{j} + xy\underline{k} = \underline{i}\,\partial\phi/\partial x + \underline{j}\,\partial\phi/\partial y + \underline{k}\,\partial\phi/\partial z$

$\frac{\partial\phi}{\partial x} = yz$ ——————— (1)

$\frac{\partial\phi}{\partial y} = y^2 + xz$ ——————— (2)

$\frac{\partial\phi}{\partial z} = xy$ ——————— (3)

(1) $\Rightarrow \phi = xyz + f(y,z)$

$\frac{\partial\phi}{\partial y} = xz + \partial/\partial y\, f(y,z)$

(2) $\Rightarrow y^2 + xz = xz + \partial/\partial y\, f(y,z)$

$y^2 = \partial/\partial y\, f(y,z)$

$f(y,z) = y^3/3 + g(z)$

$\phi = xyz + y^3/3 + g(z)$

$\frac{\partial\phi}{\partial z} = xy + \frac{\partial}{\partial z}g(z)$

(3) $\Rightarrow xy = xy + \frac{\partial}{\partial z}g(z)$

$\frac{\partial}{\partial z}g(z) = 0$

$\Rightarrow g(z) = c$

↑

Constant

f at (1, 3, 0)

directional derivative $= \underline{i} + 2\underline{j} - \underline{k}$

$\underline{D}_n \phi = \underline{\nabla}\phi \cdot \underline{n}$

$\underline{n} = \dfrac{\underline{i} + 2\underline{j} - \underline{k}}{\sqrt{6}}$

$\underline{\nabla}\phi \Big|_{(1,3,0)} = 3\underline{k}$

$\underline{\nabla}\phi \cdot \underline{n} = \dfrac{-3}{\sqrt{6}}$

3. $T(x, y, z) = \dfrac{80}{(1 + x^2 + 2y^2 + 3z^2)}$

Temperature increases in the direction of $\underline{\nabla}$ T.

$\underline{\nabla} T = \dfrac{\partial T}{\partial x}\underline{i} + \dfrac{\partial T}{\partial y}\underline{j} + \dfrac{\partial T}{\partial z}\underline{k}$

$= \dfrac{80.(-2x).i}{(1 + x^2 + 2y^2 + 3z^2)^2} + \dfrac{80(-4y)j}{(1 + x^2 + 2y^2 + 3z^2)^2} + \dfrac{80(-6z)\underline{k}}{(1 + x^2 + 2y^2 + 3z)^2}$

$\underline{\nabla} T \Big|_{(1,1,-2)} = -\dfrac{160}{16^2}\underline{i} - \dfrac{320}{16^2}\underline{j} + \dfrac{12 \times 80}{16^2}\underline{k}$

$\underline{\nabla} T \Big|_{(1,1,2)} = -\dfrac{10}{16}\underline{i} - \dfrac{20}{16}\underline{j} + \dfrac{12 \times 5}{16}\underline{k}$

$= -\dfrac{5}{8}\underline{i} - \dfrac{5}{4}\underline{j} + \dfrac{15}{4}\underline{k}$

maximum rate of increase $|\underline{\nabla}T| = \dfrac{\sqrt{25 + 100 + 900}}{8}$

$= \dfrac{\sqrt{1025}}{8}$

4. $f(x, y) = 2x^3 - 6x^2y + 18y^3 + 3x^2$

$\dfrac{\partial f}{\partial x} = 6x^2 - 12xy + 6x$ $\qquad$ $\dfrac{\partial f}{\partial y} = -6x^2 + 54y^2$

$\dfrac{\partial^2 f}{\partial x^2} = 12x - 12y + 6$ $\qquad$ $\dfrac{\partial^2 f}{\partial y^2} = 108y$

$\dfrac{\partial^2 f}{\partial x \, \partial y} = -12x$

For critical points

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$6x^2 - 12x + 6x = 0$

$x(x - 2y + 1) = 0$

$x = 0$ or $x = -1 + 2y$

$-6x^2 + 54y^2 = 0$

$9y^2 = x^2$

$3y = \pm x$

$y = x/3$ or $y = -x/3$

Critical points $(0, 0)$ $\left(-3/5, 1/5\right)$ $(-3, -1)$

| Point | $A=\frac{\partial^2 f}{\partial x^2}$ | $B=\frac{\partial f}{\partial x \partial y}$ | $C=\frac{\partial^2 f}{\partial y^2}$ | $D=AC-B^2$ | Nature |
|---|---|---|---|---|---|
| (0, 0) | 6 | 0 | 0 | 0 | May be a Max, Min or Saddle |
| (-3, -1) | -18 | 36 | -108 | (+) | Max |
| $\left(-3/5, 1/5\right)$ | -3.6 | 7.2 | 21.6 | (-) | Saddle |

When D = 0

Around (0, 0)

[In a neighborhood of (0, 0)]

Check the behavior of the function

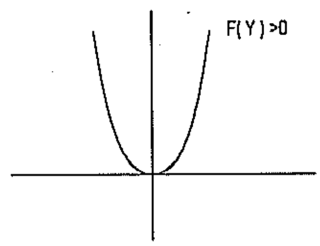
$f(x, y) = 2x^3 - 6x^2y + 18y^3 + 3x^2$

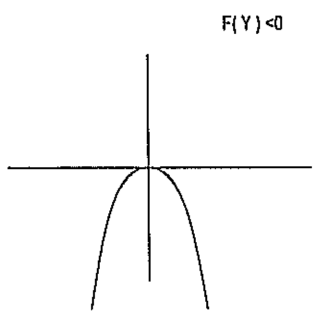
$f(0, y) = 18y^3$

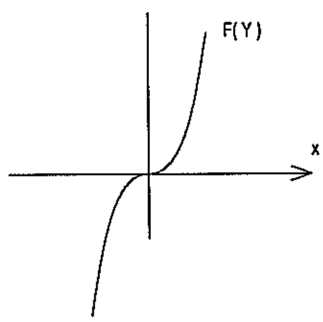
$y > 0 \quad f(0, y) > 0$

$y < 0 \quad f(0, y) < 0$

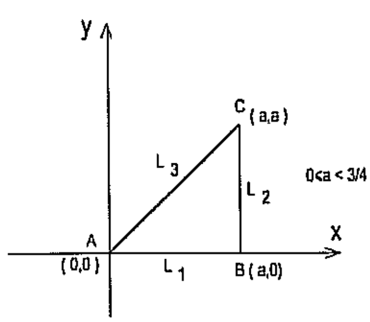
(0, 0) is a saddle point




(ii).



Find Max on the boundary.

<u>On AB</u>

$y = 0 \quad 0 \le x \le \frac{3}{4}$

$z = f(x, 0) = 2x^3 + 3x^2$

$f(x, 0) = 0$

$6x^2 + 6x = 0$

$x(x + 1) = 0 \quad \Rightarrow$ Stationary point

since $0 < a < \frac{3}{4}$

(0, 0) is the stationary point

Function value of the stationary points and at the end points

$$\left.\begin{array}{ll} \text{Stationary point} & f(0,0) = 0 \\ \text{end point} & f(0,0) = 2a^3 + 3a^2 \end{array}\right\} AB$$

<u>On BC</u>

$x = a \quad 0 \le y \le a$

$f(a,y) = 2a^3 - 6a^2y + 18y^3 + 3a^2$

$f'(a, y) = 0$

$y = \pm \frac{a}{3}$

$\left(a, \frac{a}{3}\right)$ is the stationary point

Function value $f(a, \frac{a}{3}) \quad = 2a^3 - 2a^3 + \dfrac{2a^3}{3} + 3a^2$

Stationary point $\quad = \dfrac{9a^2 + 2a^3}{3}$

$= 2a^3 - 6a^3 + 18a^3 + 3a^2$

$= 14a^3 + 3a^2$

<u>On AC</u>

$0 \le x \le a \qquad 0 \le y \le a$

$x = y$

$f(x, x) = 14x^3 + 3x^2$

$f'(x, x) = 14 \times 3x^2 + 6x$

$f'(x, x) = 0$ for stationary points

$7x^2 + x = 0$

$x = 0, \ x = -\frac{1}{7}$

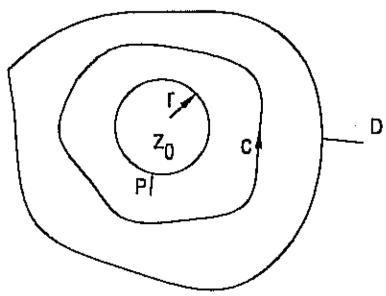
$\Rightarrow y = 0, \ y = -\frac{1}{7}$

$$\left.\begin{array}{l} \text{stationary point } (0, 0) \\ \text{end point } (a, a) \end{array}\right\} \left(-\frac{1}{7}, -\frac{1}{7}\right) \text{ already considered}$$

$f(x, y) = f(a, a) = 14a^3 + 3a^2 \longleftarrow$ max

$[\because\ 14a^3 + 3a^2 > 2a^3 + 3a^2]$
$f_{(X,Y)} = 5a^2$
$14a^3 + 3a^2 = 5a^2$
$7a^3 = a^2$
$a^2 (7a - 1) = 0$
$a = 0$ or $a = 1/7$
$a > 0$ (given)
$\therefore\ a = 1/7$

5. Let f(z) be analytic in D and $z_0$ is a fixed point in D the fu$^n$ $\frac{f(z)}{z - z_0}$ is not analytic in D and the integral $\oint \frac{f(z)}{z - z_0}$ need not to be zero.




Proof

$$\oint \frac{f(z)}{z - z_0} dz = \oint_{\Gamma} \frac{f(z)}{z - z_0} dz$$

Where Γ is a circle with radius r
Centred at $z_0$

$\Gamma : z : z_0 + re^{i\theta}, \theta :\ 0 \to 2\pi$
$z = z_0 = re^{i\theta}$ and $dz = i\, re^{i\theta}\, d\theta$

but $\oint \dfrac{f(z_0)}{z-z_0}dz = f(z_0) \quad \oint \dfrac{1}{z-z_0} = 2\pi\iota \; f(z_0)$

since $\oint\limits_c \dfrac{f(z)}{z-z_0}dz = \oint\limits_c \dfrac{f(z_0)}{z-z_0}dz + \oint\limits_c \dfrac{f(z)-f(z_0)}{z-z_0}dz$

w.m.s.t. $\oint\limits_0 \dfrac{f(z)-f(z_0)}{z-z_0}dz = 0 \left[ \text{ie} \oint\limits_r \dfrac{f(z)-f(z_0)}{z-z_0}dz = 0 \right]$

Since f(z) in cts at $z_0$


Take any $\varepsilon > 0$, Then $\quad \delta > 0$ s.t.
$f(z) - f(z_0) \; | < \varepsilon$ when ever $| z - z_0 | < \delta$
Let $r < \delta$ and consider the contour $c_0$ centered at $z_0$ with radius r.

Thus $\left| \oint\limits_r \dfrac{f(z)-f(z_0)}{z-z_o} \right| < \dfrac{\epsilon}{2\pi r} . \underset{\uparrow}{2\pi r}$

length of $C_0$

$< \epsilon$

The value of ε can be made arbitrary small

$\therefore \oint\limits_r \dfrac{f(z) - f(z_0)}{z-z_0}dz = 0$

Hence $\oint\limits_c \dfrac{f(z)}{z-z_0}dz = 2\pi\iota \; f(z_0)$

$f(z_0) = \dfrac{1}{2\pi\iota} \oint\limits_c \dfrac{f(z)}{z-z_0}dz$

(1) $f(z) = e^z$
f(z) is analytic wherever in the domain D
$z = 2$ is a singular point
$z = 2$ is outside of the domain

$\therefore$ using cauchy's integral formulae $\dfrac{1}{2\pi\iota} . \oint \dfrac{e^z}{z-2}dz = 0$

Then 2 is outside of $| z | = 1$ and inside of $| z | = 3$

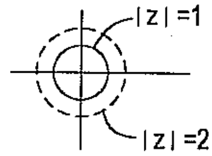
i. $C : | z | = 1$ , Then $\dfrac{e^z}{z-2}$ is analytic in c :

$\int\limits_{|z|=1} \dfrac{e^z}{z-2}dz = 0$

ii. c : $|z| = 3$, then $\dfrac{e^z}{z-2}$ is not analytic, then by C.I.F, $f(z) = e^z$ , $z_0 = 2$

$$\therefore f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}dz$$

$$\Rightarrow f(2) = \frac{1}{2\pi i}\oint \frac{e^z}{z-2}dz = e^2$$




6. (i). $f(z) = \sum_{n=0}^{\alpha} \dfrac{f^n(0)}{n!}(z-0)^n$

$f(z) = e^z, f^1(z) = e^z$ , ..................... $f^n(z) = e^z$

$\therefore\ f^n(0) = 1$

$$\therefore f(z) = e^z = \sum_{n=0}^{\alpha} \frac{1}{n!} z^n$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + .......... < 0$$

(ii). $\sum_{x=0}^{\alpha} z^n = \dfrac{1}{1-z}$ $\qquad \left|2/z_1\right| < 1$

Let $z = z_1 - 1$ $\qquad |z| > 2$

$|z_1 - 1| > 2$

$$\therefore f(z) = -\frac{z_1}{z_1-2} = \frac{-1}{\left(1-2/z_1\right)} = -\sum_{n=0}^{\alpha}\left(2/z_1\right)^n$$

$$f(z) = -\sum_{n=0}^{\alpha}\left(\frac{2}{z-1}\right)^n$$

(b). for $|z| < 1$

$$f(z) = \frac{1}{1+z}$$

$$= \sum_{n=0}^{\alpha} \frac{f^n(0)}{n!}(z-1)^n$$

$$f^1(z) = -\frac{1}{(1+z)^2}$$

$$f^2(z) = \frac{2}{(1+z)^3}.........f^n(z) = \frac{n}{(1+z)^{n+1}}$$

$\dfrac{1}{1+z} = \sum_{n=0}^{\alpha} (-1)^n z^n$, $|z| < 1$ analytic & differentiable

Since $\int \frac{1}{1+z} dz = \sum_{n=0}^{\alpha} (-1)^n \int z^n dz$

$= \sum_{n=0}^{\alpha} (-1)^n \frac{z^{n+1}}{n+1}$

$= \sum_{n=1}^{\alpha} (-1)^n \frac{z^n}{n}$