MPZ 4230 – Engineering Mathematics II
Assignment No. 03 – Academic Year 2006

1. (i). Prove that the fourier series of the function which is equal to x^2 in $(-\pi, \pi)$ is

$$\frac{\pi^2}{3}+4\left\{-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+.........+(-1)^n\frac{\cos nx}{n^2}+........\right\}$$

Deduce that , $\frac{1}{1^2}+\frac{1}{2^2}+.........+\frac{1}{n^2}+.........=\frac{\pi^2}{6}$

(ii). Prove that the Fourier series of the x^2 in $(0, 2\pi)$ is $\frac{4\pi^2}{3}+\sum_{n=1}^{\alpha}\left(\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right)$

Calculate the sum of this series for x = 0

2. Find the Fourier series of function $f(x) = x + x^2$ for $-\pi < x < \pi$, Hence show that

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+..........=\frac{\pi^2}{6}$

(ii). $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}............=\frac{\pi^2}{8}$

3. Determine the Fourier expansion of the periodic function whose definition in one period is

$$f(x)=\begin{cases}2 & 0<x<\frac{2\pi}{3}\\ 1 & \frac{2\pi}{3}<x<\frac{4\pi}{3}\\ 0 & \frac{4\pi}{3}<x<2\pi\end{cases}$$

4. Obtain a Fourier series for the function $f(x) = |\sin x|$ where $-\pi < x < \pi$
Sketch the graph

5. If $f(x) = \cos \mu x$ in $-\pi \le x \le \pi$ (μ not an integer, show that,

$$f(x)=\frac{2\mu}{\pi}\sin\mu\pi\left[\frac{1}{2\mu^3}+\frac{\cos x}{1-\mu^2}-\frac{\cos 2x}{2^2-\mu^2}+........+\frac{(-1)^{n-1}\cos nx}{n^2-\mu^2}+.......\right]$$

Deduce that

$$\cot\mu\pi=\frac{2\mu}{\pi}\left[\frac{1}{2\mu^2}+\frac{1}{\mu^2-1^2}+\frac{1}{\mu^2-2^2}+........+\frac{1}{\mu^2-n^2}+.......\right]$$

Hence show that $\sum_{n=1}^{\alpha}\frac{1}{9n^2-1}=\frac{1}{2}-\frac{\pi\sqrt{3}}{18}$

6. Expand $f(x) = |\cos x|$ in a Fourier series in the interval $(-\pi, \pi)$

Please send your answers on or before 2006 to the following address
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MPZ 4230 – Engineering Mathematics II
Model Answers-03 – Academic year 2006

(1) (i) $\frac{a_o}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \underbrace{\cos nx}_{d\frac{(\sin nx)}{n}} dx$$

$$= \frac{1}{\pi}\left[x^2 . \left.\frac{\sin nx}{n}\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin nx}{n}.2x\,dx\right]$$

$$= -\frac{1}{\pi}\left\{\frac{2}{n}\int_{-\pi}^{\pi} \alpha \sin nx\,dx\right.$$

$$= -\frac{2}{\pi n}\left[\frac{\sin nx}{n^2} - \frac{x}{n}\cos nx\right]_{-\pi}^{\pi} = +\frac{2}{n\pi}\left[\frac{2\pi}{n}\cos n\pi\right]$$

$$= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

Since x^2 is an even function $b_n = 0$

$$\therefore f(x) = \frac{\pi^2}{3} + \frac{4}{n^2}(-1)^n\{\cos nx\}$$

$$x^2 = \frac{\pi^2}{3} + 4\left\{-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + ------ + \frac{(-1)^n}{n^2}\cos nx + ---\right\}$$

Putting $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\left\{\frac{1}{12} + \frac{1}{2^2} + --- + \frac{1}{n^2} + ---\right\}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + --- + \frac{1}{n^2} + --$$

$$\frac{a_o}{2} = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{8\pi^2}{6} = \frac{4}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx = \frac{2}{\pi n^2}[x\cos nx]_0^{2\pi}$$

$$= \frac{4}{n^2}[1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \underbrace{\sin nx}_{d\frac{(\cos nx)}{-n}} dx$$

$$= \frac{1}{\pi}\left[x^2 . \frac{\cos nx}{-n}\Bigg|_0^{2\pi} + \int_0^{2\pi} \frac{\cos nx}{n} .2x\, dx \right]$$

$$= \frac{1}{\pi}\left\{ \frac{4\pi^2}{-n} + \frac{2}{n}\left[\frac{\cos nx}{n^2} + \underbrace{\frac{x}{n}\sin nx}_{0} \right]_0^{2\pi} \right\}$$

$$= \frac{1}{\pi}\left[\frac{4\pi^2}{-n}\right] = -\frac{4\pi}{n}$$

$$f(x) = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx \right)$$

When $x = 0$ sum of the series

$$\underline{\underline{\frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}}}$$

(2) (i) $f(x) = x + x^2 \qquad -\pi < x < \pi$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x + x^2)dx = \frac{1}{2\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\underbrace{\cos nx\, dx}_{d\frac{(\sin nx)}{n}}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\cos nx}{n^2} + \frac{x}{n}\sin nx\right]_{-\pi}^{\pi} + \left[\frac{x^2 \sin nx}{n}\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin nx}{n}.2x\, dx\right.$$

$$= +\frac{2}{\pi n}\left\{\frac{2\pi}{n}\cos n\pi\right\} = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\sin nx\, dx = -\frac{2}{n}(-1)^n + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx$$

$$= -\frac{2(-1)^n}{n}$$

$$f(x)=\frac{\pi^2}{3}+4\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}\right\}$$

$x=\pi$

$$\pi+\pi^2=\frac{\pi^2}{3}+4\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi\right\}$$

$$\pi+\frac{2}{3}\pi^2=4\left\{\frac{1}{1^2}+\frac{1}{2^2}+---+\frac{1}{n^2}+--\right\} \quad \text{-------- (1)}$$

$x=-\pi$

$$-\pi+\frac{2}{3}\pi^2=4\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi\right\}=4\left\{\frac{1}{1^2}+\frac{1}{2^2}+---+\frac{1}{n^2}+--\right\}\text{--(2)}$$

$$(1)+(2)\quad \frac{4}{3}\pi^2=8\left\{\frac{1}{1^2}+\frac{1}{2^2}+---+\frac{1}{n^2}+--\right\}$$

$$\frac{1}{1^2}+\frac{1}{2^2}+---+\frac{1}{n^2}+--=\frac{\pi^2}{6}$$

$x=0$

$$f(x)=0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$0=\frac{\pi^2}{3}+4\left\{\frac{-1}{1^2}-\frac{1}{3^2}-\frac{1}{5^2}+---\right\}+4\left\{\frac{1}{2^2}+\frac{1}{4^2}+--\right\}\text{--(2)}$$

$$\text{From (1)}\quad \frac{2}{3}\pi^2=4\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+---\right\}+4\left\{\frac{1}{2^2}+\frac{1}{4^2}+---\right\}\text{-- (3)}$$

$(3)-(2)$

$$\pi^2=8\left\{\frac{1}{12}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right\}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+----=\frac{\pi^2}{8}$$

(3) $$a_0=\frac{1}{\pi}\left\{\int_0^{2/3\pi}2dx+\int_{2/3\pi}^{4/3\pi}1dx\right\}$$

$$=\frac{1}{\pi}\left\{\frac{4}{3}\pi+\frac{4}{3}\pi-\frac{2}{3}\pi\right\}=2$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{2/3\pi} 2\cos nx\,dx + \int_{3/3\pi}^{4/3\pi} \cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2\sin nx}{n}\Bigg|_0^{2/3\pi} + \frac{\sin nx}{n}\Bigg|_{2/3\pi}^{4/3\pi}\right\}$$

$$= \frac{1}{\pi n}\left\{2\left(\sin\frac{2}{3}n\pi\right) + \sin\left(\frac{4}{3}n\pi\right) - \sin\left(\frac{2}{3}n\pi\right)\right\}$$

$$= \frac{1}{\pi n}\left\{\sin\left(\frac{2}{3}n\pi\right) + \sin\left(\frac{4}{3}n\pi\right)\right\}$$

$$a = 1 \quad a_1 = \frac{1}{\pi}\{0\}, \quad a_n = \frac{1}{\pi n}\left\{\underbrace{\sin\left(\pi - \frac{\pi}{3}\right)^n} + \sin\left(\pi + \frac{\pi}{3}\right)^n\right\}$$

$$\sin \pi/3\, n - \sin \pi/3\, n = 0$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{2/3\pi} 2\sin nx\,dx + \int_{2/3\pi}^{4/3\pi} \sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{2\frac{\cos nx}{-n}\Bigg|_0^{2/3\pi} + \frac{\cos x}{-n}\Bigg|_{2/3\pi}^{4/3\pi}\right\}$$

$$= -\frac{1}{\pi n}\left\{2\cos\frac{2}{3}n\pi - 2 + \cos\frac{4}{3}n\pi - \cos\frac{2}{3}n\pi\right\}$$

$$= -\frac{1}{n\pi}\left\{\underbrace{\cos\frac{2}{3}n\pi}_{\{-2\cos\pi/3\, n - 2\}} + \cos\frac{4}{3}n\pi - 2\right\}$$

$$= \frac{2}{n\pi}\left\{\cos\pi/3\, n + 1\right\}$$

$$n = 1 \quad b_1 = \frac{2}{\pi}\left\{3/2\right\}$$

$$= +\frac{3}{\pi}$$

$$b_2 = \frac{1}{\pi}\left\{\frac{1}{2}\right\} \quad b_3 = \frac{2}{3\pi}\{0\} \quad b_4 = \frac{1}{2\pi}\left\{\frac{1}{2}\right\} \quad b_5 = \frac{2}{5\pi}\left\{\frac{3}{2}\right\}$$

$$f(x)=1+\frac{3}{\pi}\left\{\sin x+\frac{\sin 2x}{2}+\frac{\sin 4x}{4}+\frac{\sin 5x}{5}+---\right\}$$

(4) Function is even & period π. Consist only cosine terms

$$a_o=\frac{4}{\pi}\int_0^{\pi/2}|\sin x|dx=-\frac{4}{\pi}\cos x\Big|_0^{\pi/2}=\frac{4}{\pi}$$

$$a_n=\frac{4}{\pi}\int_0^{\pi/2}|\sin x|\cos 2nx\,dx$$

$$=\frac{4}{\pi}\cdot\frac{1}{2}\left[\int_0^{\pi/2}\sin(x+2nx)dx+\int_0^{\pi/2}\sin(x-2nx)dx\right.$$

$$=\frac{2}{\pi}\left[-\frac{1}{2n+1}\cos(2n+1)x\Big|_0^{\pi/2}+\frac{1}{2n-1}\cos(2n-1)x\Big|_0^{\pi/2}\right]$$

$$=\frac{2}{\pi}\left[\frac{1}{2n+1}-\frac{1}{2n-1}\right]=\frac{4}{\pi(4n^2-1)}$$

$$(\sin x)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$$

$$=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+----+\frac{\cos 2nx}{4n^2-1}+--\right)$$

(5) $$\frac{a_o}{2}=\frac{2}{\pi}\int_0^{\pi}\cos\mu x\,dx=\frac{2}{\pi}\left[\frac{\sin\mu x}{\mu}\right]_0^{\pi}$$

$$=\frac{2\sin\mu\pi}{\pi\mu}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\cos\mu x\cos nx\;dx=\frac{\sin\mu\pi\cos\pi}{\pi}\left(\frac{1}{n+\mu}-\frac{1}{(n-\mu)}\right)$$

$$=\frac{-2\mu\sin\mu\pi\cos n\pi}{\pi(n^2-\mu^2)}$$

When n is odd $\quad a_n = \dfrac{2\mu \sin \mu\pi}{\pi(n^2 - \mu^2)}$

When n is even $\quad a_n = \dfrac{-2\mu \sin \mu\pi}{\pi(n^2 = \mu^2)}$

$$f(x) = \frac{2\mu \sin \mu\pi}{\pi}\left\{\frac{1}{2\mu^2} + \frac{\cos x}{1-\mu^2} - \frac{\cos 2x}{2^2 - \mu^2} + --\right\}$$

$f(x) = \cos \mu x$, setting x =π in the above result we get,

$$\frac{\cos \mu\pi}{\sin \mu\pi} = \cot \mu\pi = \frac{2\mu}{\pi}\left\{\frac{1}{2\mu^2} + \sum \frac{1}{\mu^2 - n^2}\right\}$$

Putting $\mu = \frac{1}{3}$ in above result,

$$\cot\frac{\pi}{3} = \frac{2}{3\pi}\left\{\frac{9}{2} + \frac{9}{1-\left(\frac{1^2}{9}\right)} + \frac{9}{1-2^2 \times 9} + ---+ \frac{9}{1-n^2 \times 9}\right\}$$

$$\frac{1}{\sqrt{3}} = \frac{18}{3\pi}\left\{\frac{1}{2} - \sum_{n=1}^{\infty}\frac{1}{1-9n^2}\right\}$$

$$\frac{1}{2} - \frac{\sqrt{3}\pi}{18} = \sum_{n=1}^{\infty}\frac{1}{9n^2 - 1}$$

(6) Since function f(x) is an even function

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx$$

Where $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} |\cos x| dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \ dx + \int_{\pi/2}^{\pi} (-\cos x) dx\right]$$

$\left(\because \cos x \text{ is - ce when } \pi/2 < x < \pi\right)$

Hence in $\left(\pi/2, \pi\right), |\cos x| = -\cos x$

$$= \frac{2}{\pi}\left[|\sin x|_0^{\pi/2} - |\sin x|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}[(1-0)-(0-1)) = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}|\cos x|\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x \cos nx \; dx + \int_{\pi/2}^{\pi}(-\cos x)\cos nx\, dx\right.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos x \cos nx \; dx - \int_{\pi/2}^{\pi} 2\cos x \cos nx \; dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}\cos(n+1)x + \cos(n-1)x\, dx - \int_{\pi/2}^{\pi}\cos(n+1)x + \cos(n-1)x dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)^n}{n+1} + \frac{\sin(n-1)x}{n-1}\Bigg|_0^{\pi/2} - \Bigg|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\Bigg|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} + \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right\}$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right]$$

$$= \frac{2\cos n\pi/2}{\pi}\left(\frac{-2}{n^2-1}\right) - \frac{-4\cos n\pi/2}{\pi(n^2-1)} \quad (n \neq 1)$$

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos^2 x - \int_{\pi/2}^{\pi}\cos^2 x \; dx = 0\right.$$

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + ---\right]$$